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Knowledge spillovers and the timing of R&D policy

Abstract:
We analyze how knowledge spillovers influence the optimal timing of R&D policy. Using numerical simulations we find that optimal subsidies to R&D may be rising over time even when the returns to knowledge is decreasing. The optimal time profile of the subsidies is determined by the elasticity of scale in the R&D production function, which again depends on both the returns to knowledge and the returns to labor.

Keywords: Innovation policy; R&D; Technological spillovers.

JEL classification: O32; O38.

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1 Introduction

Knowledge spillovers from R&D imply that governments should support R&D activity. Decreasing returns to knowledge (Jones, 1995) might suggest that the spillovers decline over time. Moreover, the existing literature on the timing issue find that governments' support should fall over time. However, in this paper we find that optimal subsidies to R&D may be rising over time when the returns to knowledge is decreasing.

A major challenge in designing R&D policies is to determine the magnitude of the external effects associated with R&D activities (Griliches, 1995; Jones and Williams, 2000; Klette et al., 2000). One major component of the external effects consists of knowledge spillovers. By knowledge spillovers we mean that the knowledge generated from firms' R&D activity spills over to other firms and expands later R&D opportunities. These spillovers are external to the individual firms and should be targeted by policy measures, e.g. subsidies. An R&D policy aimed at correcting the external effect requires estimates of the size of the knowledge spillovers. Further, the R&D policy modeling also needs to address the timing of the government support to R&D activity. Support to R&D should change over time if the magnitude of knowledge spillovers differs between time periods.

The literature on the timing issue is rather limited. One of the few studies that analyze the change in optimal R&D subsidies over time is Perez-Sebastian (2007). Perez-Sebastian (2007) develops a growth model with imitation of foreign ideas. There is a negative externality from R&D activity stemming from reduced imitation opportunities as the economy accumulates knowledge. By numerical simulations he shows that R&D subsidies should be rising over the transition path towards the balanced growth path. However, the main driver of this result is the reduction in the imitation externality over time rather than changes in the knowledge spillovers.

There are some studies of the timing issue for subsidies to environmental technologies. The timing of optimal environmental R&D subsidies is studied by Gerlagh et al. (2008 and 2009). They develop a theoretical growth-model and find that the optimal R&D subsidy rate to abatement technologies falls over time. However, their finding is stemming from inefficiencies in the R&D market related to limited patent lifetime rather than from a change in the size of knowledge spillovers over time. Kverndokk and Rosendahl (2007) find that newly adopted technologies should be subsidized more than older technologies in a theoretical model where learning effects are analyzed. The learning effects are strongest for newly adopted technologies so that the optimal subsidies decrease over time, as the learning effects diminish. Their technology externalities, however, come from learning effects, as opposed to R&D externalities generated by optimizing R&D firms. Heggedal and Jacobsen (2008) analyze environmental R&D subsidy reforms within a computable general equilibrium model of the Norwegian economy. They find that an R&D subsidy rate decreasing over time generates the most efficient outcome. However, it is not clear whether their result both constitutes the optimal time profile and holds for R&D within other economies in general.

In this paper we study how the inefficiency caused by knowledge spillovers influence the timing of optimal R&D policy. We solve numerically an intertemporal maximization problem where the government chooses optimal R&D subsidy rates in each period. We find that an elasticity of scale larger than one in the R&D production function implies increasing subsidy rates, while an elasticity of scale smaller than one implies falling subsidies. Further, the simulations give constant optimal subsidy rates over time if the elasticity of scale is equal to one.

There are various estimates of the output elasticities of the inputs to the R&D production function, e.g. the elasticities with respect to knowledge and labor[1]. However, most studies find the elasticity of scale in the R&D production function to be larger than one, which implies that increasing R&D subsidies are optimal. Hence, we arrive at the opposite conclusion compared to the existing literature on the timing issue for subsidies to environmental technologies discussed above.

The intuition behind our results is found by considering how a marginal increase in the stock of knowledge affects future production of knowledge. First, the productivity gain from a new idea is declining in the size of the knowledge stock if the output elasticity with respect to knowledge is smaller than one, i.e. decreasing returns to new ideas (Jones, 1995 and 1999). Hence, this effect contributes to diminish the external effect from R&D conducted in later periods as the stock of knowledge grows. Second, a marginal increase in the stock of knowledge also generates an expansion of labor into research, which contributes to increase the production of knowledge. The intuition behind this effect is that the productivity of the R&D firms increase when the stock of knowledge increases. Profit maximizing R&D firms hire more workers when their productivity increases, and this contributes to expand the future production of knowledge. Hence, this effect contributes to increase the external effect of R&D as more workers will utilize the knowledge in future periods.

The total increase in future production of knowledge generated by a

[1] See Porter and Stern (2000), Gong et al. (2004), Abdih and Joutz (2005), Pessoa (2005), and Samaniego (2007) for estimates of the elasticities.

marginal increase in the stock of knowledge is constant over time when the elasticity of scale equals one, as the second effect cancel out the first effect. These effects have proven crucial for the optimal differentiation of R&D subsidies between sectors. Heggedal (2008) finds that the emerging technology with the smaller knowledge stock should be subsidized more (less) if the scale elasticity is smaller (larger) than one.

The paper is organized as follows. The model is laid out in Section 2, while Section 3 analyzes the optimal time profile of the subsidy rates to R&D using simulations. Section 4 concludes.

2 The model

In order to analyze how the externalities from knowledge spillovers affect the optimal distribution of policy incentives across time, we only model the R&D industry. We abstract from the rest of the economy in order to focus on the underinvestment in R&D caused by the knowledge spillovers. We assume that the R&D industry is relatively small compared to the whole economy, so that input and output prices are constant. Section 3.1.3 relaxes this assumption.

In the model private R&D firms enter the market in each period and produce patents without taking into account that their production of knowledge spills over to future R&D. The social planner maximizes the discounted social surplus and sets optimal subsidies for the whole time period to correct for the externality stemming from knowledge spillovers.

2.1 The Technology

The aggregate production of new patents (knowledge) in the economy is given by the type of production function that is used in endogenous growth models with horizontal innovation, e.g. Romer (1990), and with vertical innovation, e.g. Aghion and Howitt (1992):

$$X_t = A_{t-1}^{\phi} L_t^{\lambda}, \tag{1}$$

where X_t is the production of patents, L_t is the labor input, $\lambda \in [0, 1)$ is the output elasticity with respect to labor, A_{t-1} is the accumulated patents from previous periods, i.e. the knowledge stock, and ϕ is the output elasticity with respect to patents, i.e. the spillover parameter. The decreasing returns with respect to labor on an aggregate level is motivated by heterogeneous productivity between research projects in the R&D industry (see the following section). The spillover parameter reflects the effect of the existing knowledge stock on the production of new patents. We do not restrict the values ϕ can take: $\phi > 1$ gives increasing returns to knowledge and $\phi < 1$ gives decreasing returns to knowledge ($\phi < 0$ gives negative spillovers , i.e. "fishing out" effect). The elasticity of scale in R&D production equals $\phi + \lambda$.

The knowledge stock evolves according to

$$A_t = A_{t-1} + X_t = \sum_{i=0}^{t} X_i, \tag{2}$$

where X_0 is the initial knowledge stock.

2.2 The R&D industry

The firms in the R&D industry sell patents at a given price. We assume that all patents have the same value in the market, i.e. all the firms face the same patent price P.

When a patent is produced, the knowledge embedded in the patent is freely available to other firms in future periods, i.e. the knowledge stock is a public good. The firms do not take into account that their patent production influence the productivity of future R&D. We assume that there are no other externalities. Hence, the only externality in our model stems from the knowledge spillovers.

Further, in each period there is a continuum of research projects with different productivity. A high productivity project requires less labor to produce a patent than a low productivity project. The individual firms are entities with private information of one of the research projects. In each period the firms decide whether to enter the industry and sell the patent to the given price P.

There is free entry into the industry and the least productive firm to enter earns zero profit, i.e. on an industry level there are decreasing returns to labor. The firms take the wage rate w, the unit subsidy rate on labor z_t, and P as given, and firms enter the industry until

$$P\frac{\partial X_t}{\partial L_t} - (w + z_t) = 0, \tag{3}$$

where $\frac{\partial X_t}{\partial L_t} = \lambda A_{t-1}^{\phi} L_t^{\lambda-1}$ follows from (1). The free entry condition given by (3) can be solved for L_t to get the labor demand in the R&D industry:

$$L_t = \left(\frac{PA_{t-1}^{\phi}\lambda}{w-z_t}\right)^{\frac{1}{1-\lambda}} \text{ for all } t = 1, ..., t, .. \tag{4}$$

2.3 The Social planner

The social planner maximizes the social surplus SS over the infinite-horizon by setting the subsidy rate in each period. The social surplus is given by

$$SS = PS + GS, \tag{5}$$

where PS is the producer surplus, and GS is the governmental surplus.

GS, which is a cost for the government, is given by the discounted value of the R&D subsidies:

$$GS = -\sum_{t=1}^{\infty} \frac{1}{(1+r)^{t-1}} \left[z_t L_t \right], \tag{6}$$

where r is the discount rate, i.e. the interest rate. PS is given by the discounted value of the private R&D firms' profits:

$$PS = \sum_{t=1}^{\infty} \frac{1}{(1+r)^{t-1}} \left(PX_t + z_t L_t - wL_t \right). \tag{7}$$

The second term in equation (7) equals GS from equation (6), implying that the producers' extra revenue from subsidies equals the governmental cost of the R&D subsidies. Then the social planner problem is reduced to maximizing the present value of patent production:

$$\begin{aligned} &\max_{\{z_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \frac{1}{(1+r)^{t-1}} PX_t - wL_t \\ &s.t.\, L_t(z_t) > 0, (1), (2) \text{ and } (4). \end{aligned} \tag{8}$$

To solve this planner problem analytically to find the optimal time profile of the subsidy rates proves to be difficult due to the complexity of the dynamics in the problem. Rather, we solve this planner problem by numerical simulations in the following section.

Before we move on to simulations, we illustrate by a simple example that the scale elasticity is crucial for the development of the knowledge spillovers over time. Consider the unregulated equilibrium where there are no subsidies, i.e. $z_t = 0$. Rearranging the first order condition from equation (4) when $z_t = 0$ and using (1) gives the production of patents in the unregulated equilibrium as a function of the knowledge stock, A_{t-1}, and the parameters P, λ, w and ϕ:

$$X_t = \left(\frac{P\lambda}{w} \right)^{\frac{\lambda}{1-\lambda}} A_{t-1}^{\frac{\phi}{1-\lambda}}. \tag{9}$$

We see that the production function (9) is homogenous of degree $\frac{\phi}{1-\lambda}$. If $\phi + \lambda = 1$, then the production function is homogenous of degree 1, implying constant returns to knowledge on production, i.e. constant spillovers over time. If $\phi + \lambda < 1$ ($\phi + \lambda > 1$), then there is decreasing (increasing) returns to knowledge on production, and the spillovers decrease (increase) over time.

Intuition behind the effect of the scale elasticity on the spillovers can be gained by noticing that there are two opposing effects from a larger knowledge stock on the patent production. First, the labor input grows as R&D productivity improves when the knowledge stock grows. This effect gives that the patent production is increasing in the knowledge stock. Second, there is less productivity gain from new knowledge when the knowledge stock is large. This effect gives that the patent production is decreasing in the knowledge stock. The scale elasticity determines which effect is dominating in the unregulated equilibrium.

That the scale elasticity determines whether the spillovers increase or decrease over time in the unregulated equilibrium may suggest that the scale elasticity is also important for the timing of the optimal subsidies. However, since the optimal subsidies depend on the knowledge spillovers to all future periods, and *vice versa*, we cannot calculate how the optimal subsidies develop over time. Thus, we proceed to analyze the optimal subsidies by numerical simulations.

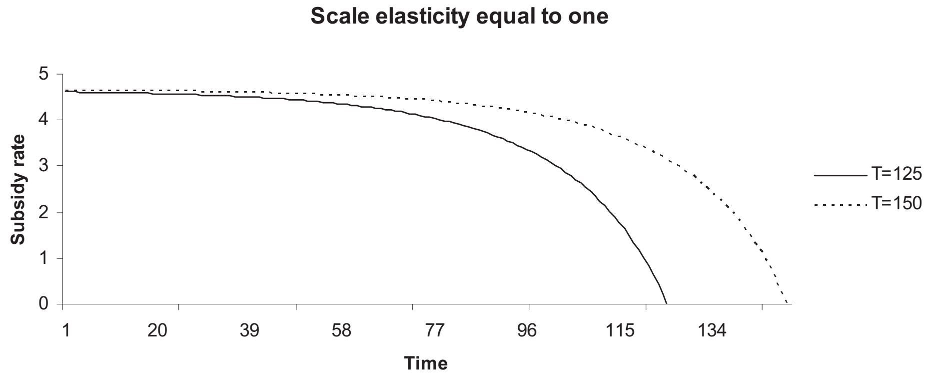
3 Numerical analysis

To concentrate on the spillovers and the subsidy profiles, we assume that all exogenous variables and parameters are constant over time. To make the results following from different combinations of $\phi+\lambda$ comparable, we adjust the wage rate, w, so that the production level in the unregulated equilibrium, i.e. the production when there are no subsidies, in the first period is the same in all scenarios. The numerical solver sets the subsidy rates for all time periods simultaneously, and finds the combination of subsidy rates that maximizes the social surplus. See appendix A for a parameter list.

First we run 3 scenarios where we show the optimal subsidy rates for different scale properties. To make sure that the results from the simulation model are not affected by its finite horizon, we have run the model for different horizon lengths. Since there are no spillovers after the last period of the simulations, the optimal subsidy rates always go to zero at the end of the horizon, so we focus on the subsidy rates in the earlier periods of the simulations. By varying the horizon length, we find the length of which the simulated subsidy rates in the early periods do not change. In figure 1 the subsidy rates for two different horizon lengths when the scale elasticity equals one are seen[2].

[2] See appendix A for the parameter values used in the figures.

Figure 1



The subsidy profiles in the first 20 periods are approximately unaltered when the number of simulation periods exceeds 125, suggesting that the first 20 periods mimics the solution to the infinite horizon problem. Since we are interested in the solutions to the infinite problem, we only study the first 20 periods of the simulations. The subsidy rate is approximately constant in the first 20 periods, implying that a scale elasticity equal to one is sufficient for constant subsidies. We have run the simulation model for different parameter values in the case of a scale elasticity equal to one, and the subsidy profiles are always constant. This suggests that the spillovers in optimum follows the same dynamics as the spillovers in the unregulated equilibrium, so that the two opposing effects of increasing knowledge stock, increasing labor input due to higher labor productivity and less productivity gain from new knowledge when the knowledge stock is large, just offset each other when the scale elasticity equals one.

Figure 2 and figure 3 show the optimal subsidy rates the first 20 periods of the simulations for scale elasticities in the R&D production function larger and lower than one.

Figure 2

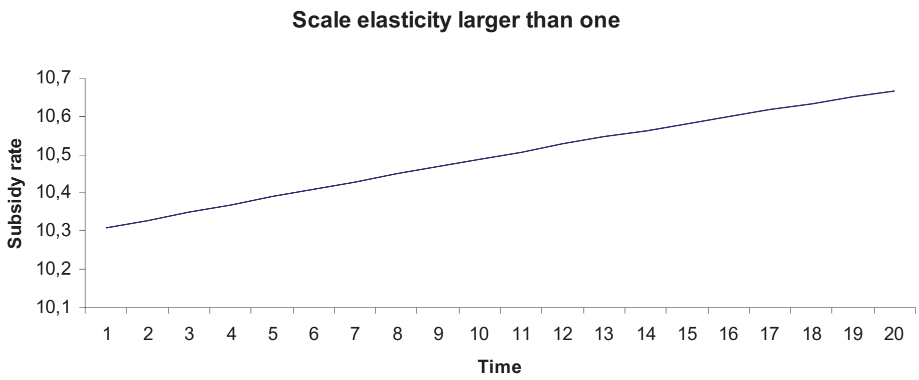


Figure 3

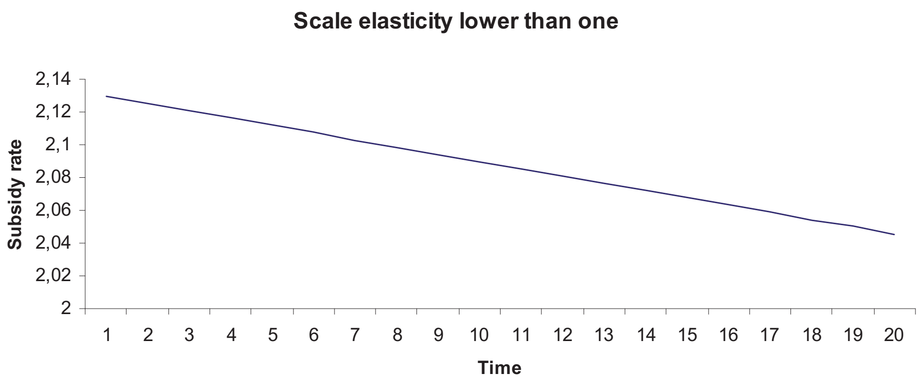


Figure 2 and figure 3 show that the optimal subsidy rate is increasing when the scale elasticity in the R&D production function is larger than one, and that the optimal subsidy rate is decreasing when the scale elasticity is lower than one. The simulations suggest that the mechanisms from the simplified example with no subsidies apply to the optimal solutions with subsidies as well. The intuition behind this result follows from that the scale elasticity determines the knowledge spillovers through the two opposing effects mentioned above. First, the spillovers are increasing in the knowledge stock because the labor input is large in late periods due to high marginal labor productivity. Second, the spillovers are declining in the knowledge stock because there is less productivity gain from new knowledge when the stock is large. These simulations do not prove the relation between the scale elasticity and the subsidy rate profile. However, we have tested for different parameter values, and all simulations give the same results with respect to the scale elasticity and the time profile of the subsidy rate.

Also for these scenarios we have run several sensitivity analyses, varying the values of the different parameters. The results have shown robust to these tests.

3.1 Extensions.

In the following we do separate 3 extensions and study how these affect the optimal subsidy rates. First we introduce a constraint on the subsidy amounts. Then depreciation of knowledge is introduced into the model, making non-increasing knowledge stocks possible. Finally, we introduce endogenous wages into the model.

3.1.1 Governmental budget constraint

Governmental subsidization has to be financed by taxes. To represent the costs of taxation, we introduce a constraint to the subsidy amounts so that the social planner has to decide between subsidizing now and subsidizing later. We study the case of a governmental budget constraint by simulating the previous scenarios with increasing knowledge stocks, and constraining the present values of the subsidies to equal 50 % of the present values of the corresponding unconstrained scenarios. The simulated subsidy profiles are unaltered when introducing a budget constraint on the subsidies, implying that the governmental budget constraint only has a level effect, and not a profile effect, on the subsidy rates.

3.1.2 Non-increasing knowledge stock

If one interprets the knowledge stock in our model as only the knowledge that is contributing directly to the production process, a non-increasing knowledge stock might not be unrealistic. To allow for this, we expand our numerical model to include depreciation of knowledge, represented by a constant depreciation rate. The function for the knowledge stock is updated to

$$A_t = (1 - \delta)A_{t-1} + X_t,$$

so that

$$A_t = \sum_{i=0}^{t} (1 - \delta)^{t-i} X_i,$$

where δ is the depreciation rate. The simulated subsidy rates are constant over time for all scale elasticities if the depreciation rate keeps the knowledge stock constant over time. The intuition is that the economic environment is the same for all periods when the knowledge stock is constant, i.e. without subsidies the private firms produce the same output in all periods. This implies that the undersupply of R&D is the same in all periods. Thus, the optimal subsidy rate is constant[3].

In the decreasing knowledge stock scenario the connection between the subsidy rates and the scale elasticity is reversed from the scenario with an increasing knowledge stock. The reason is that the dynamics of the social planner's maximization problem is reversed when the knowledge stock decreases over time compared to when the knowledge stock is increasing. Hence, the subsidy profiles in the scenario with a decreasing knowledge stock are reversed from the scenario with an increasing knowledge stock. The results stand for all the parameter values we have tested for.

3.1.3 Endogenous wage rates

A constant wage in the R&D industry, which we have assumed, may be a reasonable approximation if the R&D industry constitutes a small part of the total economy. However, if R&D's share of labor gets large, R&D wages have to increase relatively more than other industries to attract workers. The simulations are repeated with the following wage equation to allow for endogenous wages:

$$w_t = \left(\frac{L_t}{L_0}\right)^{\frac{1-\alpha}{\alpha}} w_0, \tag{10}$$

where L_0 and w_0 are base year values for labor and wage in the R&D industry, L_t and w_t are the corresponding current values, and α is the scale elasticity in an outside industry that use labor from the same pool as the R&D industry. If the elasticity of scale in the outside industry equals 1, i.e. $\alpha = 1$, there is no extra costs of using more labor in the R&D industry, so we get the same constant wage as before. The reason is that constant returns to scale in the outside industry implies a constant marginal product of labor (MPL) in that industry.

If there are decreasing returns to scale in the outside industry, i.e. $\alpha < 1$, the MPL increases for the outside industry as more labor is allocated to R&D, and the wage rate increases. Simulations for $\alpha < 1$ gives both lower optimal subsidies and optimal subsidies that are more decreasing for all scale elasticities. This means that a scale elasticity equal to one no longer is the threshold for when the time profile of the optimal subsidy rate changes from increasing to falling, i.e. it is possible to get falling subsidies whether the scale elasticity is lower, equal or larger than 1. In other words, the threshold for when the subsidy profile change sign from increasing to falling is higher the more the wage to

[3] To see this note that the maximization problem given by (8) is the same if we start from period t as if we start from period $t+n$ ($n \in (1,\infty)$), when A is constant.

R&D increases. The intuition is that higher wages, due to growing labor input in the R&D industry, reduce the expansion of labor into research in future periods following from increased productivity. This contributes to lower the increase in production in future periods, and thus reduce the knowledge spillovers.

4 Conclusion

How governments should engage in policies to spur R&D activity from private firms is an important policy question since the research markets are riddled with inefficiencies. In this paper we explore how one of these inefficiencies - externalities from knowledge spillovers - affects the distribution of policy incentives across time.

We find that the time profile of the optimal subsidy rates to R&D is determined by the elasticity of scale in the R&D production function. An elasticity of scale larger than one implies increasing optimal subsidy rates, while an elasticity of scale smaller than one implies falling optimal subsidies. Further, constant subsidy rates over time are optimal only if the elasticity of scale is equal to one.

The intuition behind this result is that the scale elasticity determines the knowledge spillovers through two opposing effects. First, the spillovers are increasing in the knowledge stock because the labor input is large in late periods due to high labor productivity. Second, the spillovers are declining in the knowledge stock because there is less productivity gain from new knowledge when the stock is large.

We also consider a second best case with a limited governmental budget for R&D subsidies. The budget constraint has no effect on the time profile of the optimal subsidy rates, only a level effect.

However, if the R&D industry is large relative to the rest of the economy, increasing wages may reduce knowledge spillovers and make the time profile of the optimal subsidy rates more falling, for given elasticity of scale.

The size of the output elasticity parameters is an empirical question. There are some studies that estimate the parameters in the aggregate R&D production function. Both Porter and Stern (2000) and Pessoa (2005) find a scale elasticity larger than one. In another study, Gong et al. (2004) find a scale elasticity smaller than one, though, the results are not significant. Another approach is Jones and Williams (2000), where the calibrated ranges of output elasticities all give an elasticity of scale larger than one. Further empirical research is needed to establish the significant ranges for the output elasticities.

In this paper we have only included one externality in the research market. Other externalities, e.g. monopoly prizing, creative destruction, and research congestion, may also influence the optimal time profile of subsidies. This is a venue for future research.

References

[1] Abdih, Y. and Joutz, F. (2005): Relating the knowledge production function to total factor productivity: an endogenous growth puzzle, IMF Working paper 05/74.

[2] Aghion, P., Howitt, P. (1992): A model of economic growth through creative destruction, Econometrica 60, 323-351.

[3] Gerlagh, R., Kverndokk, S., and Rosendahl, K.E. (2008): Linking Environmental and Innovation Policy, FEEM Working Papers 211, FEEM.

[4] Gerlagh, R., Kverndokk, S., and Rosendahl, K.E. (2009): Optimal Timing of Climate Change Policy, Environmental and Resource Economics 43, 369-390.

[5] Gong, G., Greiner, A., and Semmler, W.(2004): Endogenous Growth: Estimating the Romer Model for the US and Germany, Oxford Bulletin of Economics and Statistics 66, 147-164.

[6] Griliches, Z. (1995): R&D and Productivity: Econometric Results and Measurement Issues, in Handbook of the Economics of Innovation and Technical Change, P. Stoneman (Ed.), Blackwell, Oxford.

[7] Heggedal, T.R. (2008): On R&D and the undersupply of emerging versus mature technologies, Discussion Paper 571, Statistics Norway.

[8] Heggedal, T.R. and Jacobsen, K. (2008): Timing of innovation policies when carbon emissions are restricted: An applied general equilibrium analysis, Discussion Paper 536, Statistics Norway.

[9] Jones, C. I. (1995): R&D based models of economic growth, Journal of Political Economy 193, 759-84.

[10] Jones, C. I. (1999): Growth: With or Without Scale Effects, The American Economic Review 89, 139-144.

[11] Jones, C.I. and Williams, J.C. (2000). Too Much of a Good Thing? The Economics of Investment in R&D, Journal of Economic Growth 5, 65-85.

[12] Klette, T.J., Møen, J., and Griliches, Z. (2000): Do subsidies to commercial R&D reduce market failures? Microeconometric evaluation studies, Research Policy 29, 471-495.

[13] Kverndokk, S. and Rosendahl, K.E. (2007): Climate policies and learning by doing: Impacts and timing of technology subsidies, Resource and Energy Economics 29, 58-82.

[14] Perez-Sebastian, F. (2007): Public support to innovation and imitation in a non-scale growth model, Journal of Economic Dynamics

& Control 31, 3791-3821.
[15] Pessoa, A. (2005): "Ideas" driven growth: the OECD evidence, Portuguese Economic Journal 4, 46-67.
[16] Porter, M.E, and Stern, S. (2000): Measuring the "ideas" production function: Evidence from international patent output, Working paper 7891, National Bureau of Economic Research.
[17] Romer, P. (1990): Endogenous Technological Change, Journal of Political Economy 94, 1002-1037.
[18] Samaniego, R.M. (2007): R&D and growth: The missing link?, Macroeconomic Dynamics 11, 691-714

5 Appendix A: Parameter list

The following values were used in all simulations:

$$\begin{array}{cc} A_0 & 500000 \\ P & 1 \\ r & 0.07 \\ T & 125 \\ X_0^{UE} & 12500 \end{array},$$

UE=Unregulated equilibrium.

The tables below show the different parameter values for ϕ and λ that were simulated. The values marked with * are the values used in the figures.

Scale elasticity = 1

$$\begin{array}{c} \phi \\ \lambda \end{array} \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix} \begin{bmatrix} 0.5^* \\ 0.5^* \end{bmatrix} \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

Scale elasticity < 1

$$\begin{array}{c} \phi \\ \lambda \end{array} \begin{bmatrix} 0.05-0.09 \\ 0.9 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.25-0.6 \end{bmatrix} \begin{bmatrix} 0.1-0.54 \\ 0.45 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.1-0.25 \end{bmatrix} \begin{bmatrix} 0.35 \\ 0.35 \end{bmatrix} \begin{bmatrix} 0.4-0.55 \\ 0.1 \end{bmatrix}$$

$$\begin{array}{c} \phi \\ \lambda \end{array} \begin{bmatrix} 0.45 \\ 0.1-0.54 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.2-0.29 \end{bmatrix} \begin{bmatrix} 0.475^* \\ 0.5^* \end{bmatrix}$$

Scale elasticity > 1

$$\begin{array}{c} \phi \\ \lambda \end{array} \begin{bmatrix} 0.105 \\ 0.9 \end{bmatrix} \begin{bmatrix} 0.45 \\ 0.56-0.59 \end{bmatrix} \begin{bmatrix} 0.56-0.6 \\ 0.45 \end{bmatrix} \begin{bmatrix} 0.525^* \\ 0.5^* \end{bmatrix} \begin{bmatrix} 0.55 \\ 0.55 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.31-0.39 \end{bmatrix}$$