Discussion Papers No. 516, September 2007
Statistics Norway, Research Department

Mads Greaker and Tom-Reiel Heggedal

Lock-in and the transition to hydrogen cars
When should governments intervene?

Abstract:
The density of fuel filling stations influences consumers' utility of private car transport. Thus, to the extent that different modes of private transport require different fuels, there may exist a network externality in the consumption of private transport.

We investigate this in a formal model of the market for private transport. In the model there are two competing technologies; today's internal combustion engine based on fossil fuels, and tomorrow's hydrogen car. Due to the network externality there may exist several market equilibriums, of which one is likely to Pareto dominate the other(s). Thus, a lock-in situation is possible.

On the other hand, if either the costs of establishing hydrogen filling stations is too high or the hydrogen car technology is still in its infancy, the only equilibrium is the current internal combustion engine equilibrium. Hence, apart from internalizing the environmental externality on gasoline cars, the government has no reasons to intervene before the technology is ripe. And even then, governments should take great care so as not to create a situation of excess momentum.

Keywords: Lock-in, Path dependency, Hydrogen economy, Climate change policy

JEL classification: L11, L15, L62, Q42

Acknowledgement: We are grateful for financial support from the Renergi program of the Norwegian Research Council. We would also like to express our gratitude to Knut Einar Rosendahl, Snorre Kverndokk and Michael Hoel for all the good comments on earlier versions of this paper.

Address: Mads Greaker, Statistics Norway, Research Department. E-mail: mgr@ssb.no

Tom-Reiel Heggedal, Statistics Norway, Research Department. E-mail: tom@ssb.no

Discussion Papers comprise research papers intended for international journals or books. A preprint of a Discussion Paper may be longer and more elaborate than a standard journal article, as it may include intermediate calculations and background material etc.

Abstracts with downloadable Discussion Papers
in PDF are available on the Internet:
http://www.ssb.no
http://ideas.repec.org/s/ssb/dispap.html

For printed Discussion Papers contact:

Statistics Norway
Sales- and subscription service
NO-2225 Kongsvinger

Telephone: +47 62 88 55 00
Telefax: +47 62 88 55 95
E-mail: Salg-abonnement@ssb.no

1 Introduction

The notion "the hydrogen economy" has become a popular catchword among politicians, environmentalists and others. In a world threatened by global warming, it promises us a comfortable life in the future with plenty access to private transport that only emits water. No wonder then that governments are keen on getting "the hydrogen economy" going. For governments the relevant questions are if this at all require public support, and how and when public support should be given.

Some technology experts argue that the market entry of hydrogen cars is delayed by network externalities in the market for private transport, which has led to a *technological lock-in.* They therefore advocate subsidies for the market introduction of the hydrogen car and for infrastructure development (Melaina, 2003). The hydrogen demonstration projects, which are established through out the western world, can be seen as first steps to larger publicly financed infrastructure developments. For instance, in Scandinavia, a large part of public R&D funds are spent on buying prototype hydrogen cars from existing car manufactures and building filling stations to create a "hydrogenway" from western Norway, through Sweden and to Denmark (www.scandinavianhighway.org).

The concept of *technological lock-in* describes a situation in which some new product that presumably would increase welfare does not successfully diffuse into the market. Unruh (2000) describes what he calls the *Techno-Institutional Complex* which according to Unruh (2000) keeps profitable carbon saving technologies from being implemented. Unruh applies his concept both to the current fossil fuel based internal combustion engine (ICE) and to the whole private automobile based transportation system, and sees both as typical lock-in cases although on different levels. In a later paper Unruh (2002) argues that the escape from carbon lock-in has to be triggered from external forces, and that we cannot expect a transition to happen through "the forces of the market".

In this paper we provide a conceptual analysis of the lock-in problem with explicit point of departure in the transport market. In particular we ask the following questions: i) What factors may create a lock-in in the market for private transport in the case of hydrogen versus gasoline/diesel cars? and ii) If there is a potential lock-in in the market for personal transport, when should the government intervene?

Firstly, we find that there is no lock-in in the market for personal transport if the costs of the hydrogen technology are too high compared to the benefits of lower emissions. In such a situation, it seems better to channel public funds towards R&D projects that directly will improve the technology.

Secondly, if the hydrogen technology are just moderately more costly that the current fossil fuel technology including the carbon externality, there exists more than one possible equilibrium in the market. Moreover, the equilibriums with the lower hydrogen shares may under certain conditions be true lock-in situations. In such a case, policy makers should seek to coordinate the market to the equilibrium with high hydrogen car usage.

On the other hand, it is clearly difficult to determine when the intervention case is present. To make the right decision when to intervene in the personal transport market policy makers need substantial information on three key issues; the maturity of the hydrogen technology, the consumers benefit of making a change in technology, and the loss from unused gasoline/diesel filling capacity in the market.

There is a body of literature looking at the lock-in phenomena from a more general point of view. Katz and Shapiro (1983) look at a case in which consumers value some product more highly when it is compatible with other consumer products, for instance software programs and hardware compatibility. Further, in their model firms choose whether to make their products compatible or not with other firms' products. They find that a dominant firm may choose to remain incompatible with a rival because becoming compatible may increase the value of the rivals' products, and hence, lead to a loss in market shares.

Farrell and Saloner (1985) also analyze a general model with network externalities. Firms choose whether to switch from an old to a new technology. The decisions of the firms are modeled as a multi stage game in which one firm starts and the other firms follow sequentially. Farrell and Saloner explore different versions of this game in which firms have either complete or incomplete information about other firms' payoff functions. They define *excess intertia* to be situation in which firms do not adopt a Pareto dominant technology. This corresponds to how we define a lock-in situation. In the Farrell and Saloner model *excess intertia* cannot happen if firms have *complete information*. The result is due to the sequential timing of decisions.

In another paper Farrell and Saloner (1986) develop their ideas further, and introduce an installed base of the old technology. Due to the installed base, users of the old technology will adopt the new technology in a slow pace depending on how fast the installed base depreciate. Early adopters of the new technology must then bear the cost of a small network while waiting for more consumers to adopt the new technology. This effect can lead to *excess inertia* even with complete information. Farrell and Saloner define *excess momentum* as the inefficient adoption of a new technology. The network effect may lead to *excess momentum* as users adopt the new technology to avoid being stranded in the old technology.

In the hydrogen car case we assume that the barriers to make a hydrogen car gasoline compatible are too high to be economically viable.[1] Hence, we do not study the compatibility decisions of firms as Katz and Shapiro. Further, unlike Farell and Saloner, we look at a simultaneous game between a very large number of consumers and a large number of potential hydrogen filling station owners. Thus, we do not consider the question of whether the market will actually stay in the conventional technology or move on to the new technology.

Conrad (2005) is the only contribution that to our knowledge looks at hydrogen in the transport sector with point of departure in economic theory. Conrad models the market for traditional cars and hydrogen cars as being horizontally differentiated, and looks at the development in market shares in a dynamic model. Due to increasing gasoline prices, and investments in the network by the supplier of hydrogen cars, hydrogen cars succeed in entering the market. As opposed to this contribution, he does not model the entry decision of hydrogen filling stations and does not investigate whether there may exist more market equilibriums.

Compared to both contributions by Farrell and Saloner and the contribution by Conrad, we seek to be more explicit about what is creating the network externality in order to analyze our particular case. In this paper we combine two well known models from industrial organization, namely the Shaked and Sutton model of vertical differentiation and the Salop circle (see Shaked and Sutton (1982), Salop (1979) and also Tirole p.282 and p.296 (1997)). The result is an analytical model of lock-in in the market for private transport.

The paper is organized as follows. Section 2 defines technological lock-in, while Section 3 describes some aspects of hydrogen technology in the transport sector. The model is laid out in Section 4 and Nash equilibriums are discussed in Section 5. Section 6 analyses welfare and lock-in. Section 7 concludes.

2 What is lock-in?

Private transport is complimentary in consumption to fuel filling stations and repair workshops. The density at which these services are provided influences consumers utility of private transport. Thus, there may exist a network externality connected to the number of consumers buying a

[1] In theory a hydrogen car based on the internal combustion engine could run on both. On the other hand, presumably, all the available storage space of the car would then be occupied by either hydrogen or gasoline tanks. The more advanced hydrogen fuel cell car would be even more inconvinient to make compatible with gasoline as it would require two separate propulsion systems in the car, see below.

particular type of private transport. This is especially true for gasoline cars and hydrogen cars with respect to filling stations as a hydrogen car cannot be run on gasoline and *vice-versa*.[2]

With lock-in we are referring to a situation in which there exists two or more market equilibria, and in which the realized market equilibrium is pareto inferior to some unrealized equilibrium. This can be explained by the following simple game theoretic model. We start by looking at a simple one shot game between consumers in which the entry and exit of filling stations are not modelled explicitly. Assume that all externalities are properly internalized i.e. polluting emissions are taxed by their respective Pigouvian rates etc. The consumers choose simultaneously wether to stay in the old gasoline technology or to switch to the new hydrogen technology. Their utility depend partly on the chosen technology and partly, on the amount of filling stations in the technology they choose. Firms adjust directly according to the consumers choice so that the density of filling stations is proportional to the amount of consumers in that technology. Thus, indirectly the consumers utility depend on the other consumer's choice.

Consider two consumers with identical utility function $u = u(t_j, y_j)$, where t_j, $j = g, h$, is the chosen type of technology, and where $y_j = (1, 2)$, is the size of the network in the chosen technology. This yields the following game given in normal form:

		Player 2	
		t_g	t_h
Player 1	t_g	$u(t_g, 2), u(t_g, 2)$	$u(t_g, 1), u(t_h, 1)$
	t_h	$u(t_h, 1), u(t_g, 1)$	$u(t_h, 2), u(t_h, 2)$

The positive "indirect" network effect implies that utility is higher the more consumers have chosen the same technology *independent* of technology choice:

$$u(t_j, 1) < u(t_j, 2)$$

Then, when consumers choose technology simultaneously, there are two pure-strategy Nash equilibriums. Either both consumers stick to

[2] In theory a hydrogen car based on the internal combustion engine could run on both. On the other hand, presumably, all the available storage space of the car would then be occupied by either hydrogen or gasoline tanks. The more advanced hydrogen fuel cell car would be even more inconvinient to make compatible with gasoline as it would require two separate propulsion systems in the car.

the old technology g or both switch to the new technology h. A lock-in situation arises when $u(t_h, 2) > u(t_g, 2)$, and the consumers sticks to the old technology g. Note, however, that we need not have $u(t_h, 2) > u(t_g, 2)$ in order for there to be two equilibriums in the game.

The opposite of lock-in arises when $u(t_h, 2) < u(t_g, 2)$, and the consumers adopt the new technology h. This is still a Nash equilibrium, but it is Pareto inferior. Consumers may be afraid of getting stranded in an obsolete technology, and may make the switch even though they would be better off by staying in the old.

If we have lock-in situation, a regulator could try changing the payoffs to the consumers. This can be done either by subsidizing the consumers that switch to the new technology or by subsidizing the establishments by the firms in the new technology. Both policy options aim to make $u(t_g, 2) < u(t_h, 1)$. In the following we will analyze lock-in in a richer model which among others includes the entry and exit decisions of firms supplying complimentary goods.

3 Hydrogen technology for the transport sector

The main benefit of hydrogen as a transport fuel is that it could potentially cut all greenhouse gas emissions from the transport sector. Since the transport sector causes a large share of global carbon emissions, large scale reductions in emissions from this sector must likely happen at some point in the future if global warming is to be combatted (Caldeira et al., 2000).

Hydrogen differs from coal, oil and natural gas in that it does not exist on earth in a chemically unbound form. Rather, like electricity, its an energy carrier that must be generated using another source of energy. Thus, hydrogen will always be more expensive than the sources used to make it.

By far the cheapest way to produce hydrogen today is by steam reforming of natural gas (Service, 2004 and Padro and Putsche, 1999). On the other hand, this process consumes about 15% of the energy in the natural gas and emits CO_2, which in order for hydrogen to be a "clean fuel", must be dealt with by carbon capture and sequestration. Hydrogen can also be produced without generating CO_2 by splitting water using electricity derived from renewable energy sources like bio, wind or solar. However, this is a more expensive process due to the high cost of electricity from renewables.

In the paper we assume that hydrogen can be provided emission free to a constant unit cost. Of course, this unit cost is central in determining whether we are in lock-in situation with respect to the market for private transport or not.

At room temperature and pressure, hydrogen takes up roughly 3000 times as much space as gasoline containing the same amount of energy. This makes distributing hydrogen to hydrogen filling stations, and storing hydrogen on filling stations difficult and costly (Service, 2004). Different options exist, and among others it has been proposed that hydrogen should be produced at the filling stations either by steam reforming of natural gas or by splitting water using electricity. Alternatively, it could be distributed from centrally located production plants by pipeline and stored in special pressurized tanks at the filling station[3]. In any case distributing hydrogen will involve fixed entry costs for any filling station that wants to start to supply hydrogen. Clearly, the fixed cost accruing to the station owner is likely to be higher in a decentralized production system, than in a centralized one.

Storing hydrogen in the car is an additional challenge. Different options exist, for instance storing hydrogen as a chemical compound like a metal hydride which releases hydrogen when heated. So far most hydrogen test cars store hydrogen in pressurized tanks. Due to the low storage efficiency of such tanks, the tanks take up a lot of space in the car, and the cars have very limited driving range. Clearly, the latter aspect makes the need for a large number of hydrogen filling stations even more critical.

Lastly, hydrogen has to be converted to motion energy in the car. By far the most energy efficient way to do this is to burn hydrogen together with air in a fuel cell to generate electricity that can run an electric motor (Service, 2004). Since a hydrogen driven fuel cell car is approximately twice as energy efficient as a normal car, the fuel economy of a fuel cell car could be adequate despite hydrogen costing more than gasoline.

On the other hand, fuel cells are currently expensive (OECD, 2004). Proponents of the hydrogen economy therefore argue that we will have to start with hydrogen cars burning hydrogen as in an internal combustion engine (Cho, 2004). Such cars are only slightly more energy efficient than gasoline cars, and hence, the problems of hydrogen production costs and of storing enough hydrogen in the car is enlarged .

4 The model

The model is a one stage game between consumers and hydrogen filling stations. Consumers choose whether to buy a traditional car or a hydrogen car, and hydrogen filling stations choose whether to entry the

[3] Yet another alternative is to liquify hydrogen by cooling it to -251^0 C. Hydrogen can then be transported and stored in so called kryogenic tanks. However, the cooling process requires a considerable amount of energy, and the kryogenic tanks are voluminous due to the need for insulation.

transport fuel market and offer hydrogen to private car owners. The choice of the consumers is dependent on the density of hydrogen filling stations, while the entry decision of the filling station owners is dependent on the numbers of consumers that buy a hydrogen car.

We assume that consumers and potential filling station owners do not know the decisions of the other consumers/potential filling station owners when they must make their own choice. Further, we assume that both traditional cars and hydrogen cars are available at price equal to marginal cost. Thus, we abstract away from the functioning of the car market itself in order to focus on the potential lock-in mechanisms.

Finally, we are looking at a small country. This implies that the marginal cost of carbon emission can be set equal to the international emission quota price. Moreover, the volume in the car market we are looking at will be too small to induce any significant technological learning. Hence, the only potential lock-in mechanism is the network externality.

4.1 Hydrogen filling stations

We will use the Salop-circle in order to model the entry of hydrogen filling stations. Like in Salop (1979), the perimeter of the circle is 1, however unlike Salop, the density of consumers will be equal to the number of consumers that have bought a hydrogen car i.e. $q^h = 1 - q^g$ where q^g is the number of consumers that have bought a gasoline car.

For reasons of simplicity, we assume that the total mileage through out the life time of a car is given and equal for each consumer, and that prices for fuel are given as a price per volume unit times the life time mileage of the car. We also assume that the fixed cost of entry into the hydrogen filling station market, and the income from the sales of hydrogen are incurred in the same period.

In our model, all consumers live in a city center, and they all commute by driving along the circle around the city center. Consumers must adapt their driving pattern to the localization of filling stations, for instance, they must postpone their exit from the circle in order to refuel or they must make a stop along the circle before they would like to. Let t denote the cost per distance of driving to the nearest filling station. It should be interpreted as an inconvenience cost, that is, the more scarce the filling stations are, the more must the driving pattern be adapted.

Denote the number of hydrogen filling stations by n. We assume that the n stations are distributed uniformly along the circle. Filling station α chooses its price of hydrogen p^h_α. A consumer located at the distance $x \in \left[0, \frac{1}{n}\right]$ from filling station α will be indifferent between purchasing hydrogen from filling station α and its closest neighbor, filling station β, if:

$$p^h_\alpha + tx = p^h_\beta + t(\frac{1}{n} - x), \tag{1}$$

where p^h_β is the price of hydrogen from station β, and where $\frac{1}{n}$ is the distance between the filling stations. By solving (1) for x and multiplying with 2 since the station gets consumers from both sides along the Salop-circle, the share of consumers purchasing from station α is: $2x = \left(\frac{-p^h_\alpha + p^h_\beta + \frac{t}{n}}{t}\right)$. The demand for hydrogen facing station α can then be written: $D_\alpha(p^h_\alpha, p^h_\beta) = 2xq^h = \left(\frac{-p^h_\alpha + p^h_\beta + \frac{t}{n}}{t}\right) q^h$.

Denote the fixed cost of establishing a hydrogen filling station by f^h, and let c^h denote the cost of supplying one more hydrogen car with hydrogen for its life time mileage at the filling stations. Therefore, filling station α seeks to maximize:

$$\max_{p_\alpha} \left[(p^h_\alpha - c^h) \left(\frac{-p^h_\alpha + p^h_\beta + \frac{t}{n}}{t} \right) q^h - f^h \right].$$

Since stations are distributed uniformly along the circle, the Nash-price equilibrium is symmetric with $p^h_\alpha = p^h_\beta$. From the first-order condition for profit maximum we then obtain the price of hydrogen in the market: $p^h = c^h + \frac{t}{n}$. Note that, the more stations, the lower the mark-up. Thus, the number of hydrogen filling stations not only determines the inconvenience cost of hydrogen cars, but also the fuel costs of hydrogen cars.

Free entry requires that profit is equal to zero. We then have that: $n = \sqrt{\frac{tq^h}{f^h}}$. By inserting we get for the life time mileage price of hydrogen:

$$p^h = c^h + \sqrt{\frac{tf^h}{q^h}}. \tag{2}$$

With n filling stations for hydrogen, the average traveling distance for a consumer to a station is $\frac{1}{4n}$. This implies that the average travelling cost of reaching a hydrogen filling station will be:

$$\delta^h = \frac{t}{4n} = \sqrt{\frac{tf^h}{16q^h}}. \tag{3}$$

Note that the inconvenience cost is declining in the number of consumers that have bought a hydrogen car. This is a typical network externality. Note also that the fixed cost of entry is crucial for the density of hydrogen filling stations. We conjecture that a decentralized production system for hydrogen implies a high f^h, while centralized production system with pipeline distribution of hydrogen implies a lower f^h.

4.2 Traditional fuel filling stations

We assume that the number of traditional fuel filling stations is constant throughout the game. They entered the market at a time when hydrogen cars didn't exist. Further, it seems reasonable to assume that their entry costs are sunk, and hence, that the number of stations will not change as the number of traditional cars change (at least not in the short or intermediate term).

Ex ante the market share of traditional cars must have been 1. This yields the following values for p^g and δ^g:

$$p^g = c^g + \tau + \sqrt{tf^g},\ \delta^g = \frac{\sqrt{tf^g}}{4}, \tag{4}$$

where τ is an emission tax on traditional cars, c^g is the cost of supplying one more gasoline car with gasoline and f^g was the historical entry cost of traditional filling stations. Note that q^g does not enter 4. This is because the size of the market only matters at the time of entry, and at that time we had $q^g = 1$.

Compared to hydrogen cars, traditional cars cause carbon emissions. Since the lifetime mileage of each car is given, the lifetime emissions from a traditional car can be written as a constant e. Let σ denote the international quota price on carbon emissions. The cost of the lifetime emissions of a traditional car is then σe. We assume that the government has internalized the carbon cost i.e. $\tau = \sigma e$.

4.3 The consumers of cars

In order to model the decision of the consumers we will use the vertical differentiation model from Shaked and Sutton (1982)[4]. We assume that the car producers set price equal to the constant marginal costs, ω^g and ω^h, for a traditional and a hydrogen car, respectively. We further assume that the market is covered, that is, all consumers buy either a traditional car or a hydrogen car.

The utility of consumer x from buying a traditional car is given by:

$$U_x^g = \lambda_x \Gamma^g - \omega^g - p^g - \delta^g, \tag{5}$$

where $\lambda_x \Gamma^g$ is the gross utility of buying a traditional car, and where the other terms are the price of a traditional car, the life time mileage cost of a traditional car and the average traveling costs to the nearest traditional filling station. The parameter λ_x reflects consumer heterogeneity, that is, consumers value car quality differently i.e. some cares a lot for cars,

[4] A model with horizontal differentiation does not fundamentally change our results, see appendix A.

others are more indifferent. The parameter λ_x is uniformly distributed on $[l, m]$ with a density equal to unity, that is, $m = l + 1$.

Accordingly, the utility of consumer x from buying a hydrogen car is:

$$U_x^h = \lambda_x \Gamma^h - \omega^h - p^h - \delta^h, \tag{6}$$

where $\lambda_x \Gamma^h$ is the gross utility of buying a hydrogen car.

We assume that $\Gamma^h - \Gamma^g = \bar{\Gamma} \geq 0$, which implies that if the price, the life time mileage cost and the density of filling stations of hydrogen cars were equal to that of traditional cars, each consumer would prefer hydrogen cars. In general consumers seems to perceive *new* technologies as better than *old* technologies. [5]

We will later refer to $\bar{\Gamma}$ as the *maximum additional willingness to pay for a hydrogen car.* Moreover, from what we know about the hydrogen car technology, it seems reasonable to assume that: $\omega^h + c^h \geq \omega^g + c^g + \tau$, that is, the costs of the hydrogen car technology is higher or equal to the costs of the traditional technology including the carbon cost. Hence, for identical networks, the assumption $\bar{\Gamma} \geq 0$ ensures that hydrogen cars are able to capture any market share at all.

The marginal consumer being indifferent between a hydrogen car and a traditional car is then given by:

$$\lambda_x^* = \frac{\omega^h + p^h + \delta^h - \omega^g - p^g - \delta^g}{\bar{\Gamma}}.$$

Since λ_x is uniformly distributed on $[l, m]$, positive demand for hydrogen cars is ensured if $m\bar{\Gamma} > \omega^h + p^h + \delta^h - \omega^g - p^g - \delta^g$. This is not necessarily the case, but it puts an upper bound on the total costs of a hydrogen car for there to be any consumers interested in buying such a car. We make two more assumptions to ensure that the demand for cars is complete. Firstly, we assume $l\Gamma^g > \omega^g + p^g + \delta^g$ such that all consumers will buy one car. Secondly, we assume $\frac{\omega^h + p^h + \delta^h - \omega^g - p^g - \delta^g}{\bar{\Gamma}} > l$, and hence, there will always be a positive demand for gasoline cars.

The demand D facing the two types of cars can then be written:

$$q^i = \begin{cases} \frac{\omega^h + p^h + \delta^h - \omega^g - p^g - \delta^g}{\bar{\Gamma}} - l & \text{for traditional cars} \\ m - \frac{\omega^h + p^h + \delta^h - \omega^g - p^g - \delta^g}{\bar{\Gamma}} & \text{for hydrogen cars} \end{cases}, \ i = g, h. \tag{7}$$

[5] The reason the the percieved higher quality may also be a warm glow from contributing to a better environment. According to Andreoni (1990) voluntary contributions to a public good like the environment can be explained by what he coins *impure altruism.* In the impure altruism case the consumer gets utility from both giving (referred to as *warm glow* by Andreoni (1990)) and from the public good in question.

5 Nash equilibrium in the personal transport game

It is well known that there is excessive entry in the basic Salop model (Tirole, 1997). Hence, the Nash equilibriums we are studying are likely not Pareto optimal from a first best point of view. In the paper we compare two different second best allocations, and to the extent one of the allocations Pareto dominates the other, we might have a lock-in situation according to our definition. However, this does not imply that there does not exist another allocation that Pareto dominates both Nash equilibrium allocations.

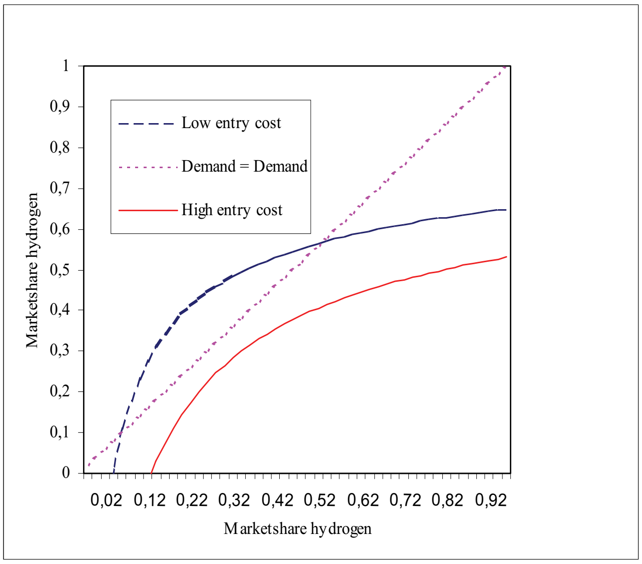
By inserting from (2), (3) and (4), we obtain : $p^h + \delta^h - p^g - \delta^g = c^h - c^g - \tau + \frac{5\sqrt{t}}{4}\left[\frac{\sqrt{f^h}}{\sqrt{q^h}} - \sqrt{f^g}\right]$. Denote $\Delta\omega = \omega^h - \omega^g$, $\Delta c = c^h - c^g$, and simplify be normalizing $f^g = 1$. The parameter f^h then directly measures the fixed cost of a hydrogen filling station in terms of the fixed cost of a traditional filling station. For the market share of hydrogen cars we get:

$$q^h = m - \frac{\Delta\omega + \Delta c - \tau}{\bar{\Gamma}} - \frac{5\sqrt{t}}{4\bar{\Gamma}}\left(\frac{\sqrt{f^h}}{\sqrt{q^h}} - 1\right). \tag{8}$$

When equation (8) is fulfilled, we have a Nash equilibrium. That is, no consumer regrets buying/not buying a hydrogen car given the number of hydrogen filling stations, and no filling station owner/potential filling station owner regrets the decision to enter/not enter the hydrogen filling station market given the number of consumers with hydrogen cars. From this definition of a Nash-equilibrium in the personal transport game, it is also easy to see that $q^h = 0$ is a Nash-equilibrium.

Since q^h is the only unknown variable in (8), we can solve for the market share of hydrogen cars directly. The market share of traditional cars is simply $1-q^h$. Equation (8) can also be plotted for different values of the parameters. Figure 1 shows two examples:

Figure 1 "Nash-equilibriums in the market for personal transport"



The diagonal coined "demand = demand" is just a help line. The two other curved lines are plots of (8) with low and high entry cost for hydrogen filling stations, respectively. The points in which the upper curved line crosses the diagonal are Nash equilibriums.

Zero hydrogen car sales is always a Nash equilibrium. Furthermore, in the low entry cost case we may either have an equilibrium with low sales of hydrogen cars, or an equilibrium with high sales of hydrogen cars. From the theory itself it is impossible to answer the question of what equilibrium will be realized. Furthermore, *a priori* all equilibriums may qualify as a lock-in. For instance, we are in the low hydrogen car sales equilibrium, and it is Pareto dominated by the high hydrogen car sales equilibrium, or we are in the high hydrogen car sales equilibrium, and it is Pareto dominated by the low hydrogen car sales equilibrium.

In the high entry cost case, the curved line never crosses the help line. Hence, there are no Nash-equilibriums with a positive demand for hydrogen cars. This happens even though the environmental externality of traditional cars is internalized through the tax τ. Thus, according to our definition, we cannot have lock-in in this case.

The insight obtained from Figure 1 can be given a formal treatment. Let $z = \sqrt{q^h}$. We then have for equation (8):

$$z^3 - \frac{m\bar{\Gamma} - \Delta\omega - \Delta c + \tau + \frac{5}{4}\sqrt{t}}{\bar{\Gamma}} z + \frac{5\sqrt{tf^h}}{4\bar{\Gamma}} = 0, \qquad (9)$$

which is a cubic equation. In order to solve the cubic equation, define $a = -\frac{m\bar{\Gamma}-\Delta\omega-\Delta c+\tau+\frac{5}{4}\sqrt{t}}{\bar{\Gamma}}$, $b = \frac{5\sqrt{tf^h}}{4\bar{\Gamma}}$ and $\Theta = \frac{b^2}{4} + \frac{a^3}{27}$. By inserting we have $\Theta = \frac{1}{\bar{\Gamma}^2}\left[\frac{25tf^h}{64} - \frac{\left(m\bar{\Gamma}-\Delta\omega-\Delta c+\tau+\frac{5}{4}\sqrt{t}\right)^3}{27\bar{\Gamma}}\right]$. If $\Theta > 0$, there is one real root and if $\Theta < 0$, there are three real roots (Turnbull, 1939). (We do not go into the special case in which $\Theta = 0$.)

Consider first the case $\Theta > 0$. The single real root is then given by the formula of Cardan: $z = \sqrt[3]{-\frac{b}{2}+\sqrt{\Theta}} + \sqrt[3]{-\frac{b}{2}-\sqrt{\Theta}}$ (Turnbull, 1939). Observe that $b > 0$ in our case. Hence, we must have $z < 0$, since $\left|\sqrt[3]{-\sqrt{\Theta}-\frac{b}{2}}\right| > \left|\sqrt[3]{\sqrt{\Theta}-\frac{b}{2}}\right|$, and $\sqrt[3]{-\sqrt{\Theta}-\frac{b}{2}} < 0$. A solution that implies $z < 0$ is not feasible[6]. Consequently, the market share of hydrogen cars will be a corner solution; $q^h = 0$. Clearly, this is the unique Nash-equilibrium when $\Theta > 0$. We have the following proposition:

Proposition 1 *There is only one Nash-equilibrium, $q^h = 0$, and thus no lock-in the market for personal transport if:*

1. *The fixed entry cost of hydrogen filling stations f^h is high*
2. *The additional cost of a hydrogen car $\Delta\omega$ and/or the cost of hydrogen supply Δc is high compared to the maximum additional willingness to pay for a hydrogen car $\bar{\Gamma}$ and the environmental tax τ*

Proof. If $\frac{25tf^h}{64} > \frac{\left(m\bar{\Gamma}-\Delta\omega-\Delta c+\tau+\frac{5}{4}\sqrt{t}\right)^3}{27\bar{\Gamma}}$, then $\Theta > 0$ and there is no feasible solution to equation (8). This can happen either if f^h is large, and/or if $\Delta\omega + \Delta c$ is large compared to $m\bar{\Gamma} + \tau$. ■

We note that the state of the technology of hydrogen cars is crucial for whether a lock-in exists. If the fixed cost of a filling station is high, it will not only be expensive to build a hydrogen distribution network, but few filling stations will also establish. The number may simply be too small for all possible values of hydrogen car sales such that a Nash-equilibrium involving $q^h > 0$ does not exist. Thus, high fixed costs of building a hydrogen filling station network is not an argument for the existence of technological lock-in.

Secondly, if the cost of a hydrogen car and the cost of hydrogen itself is too high, few people will want a hydrogen car even if they in general see hydrogen cars as higher quality. Hence, as usual, total cost of usage

[6]The negative solution to $\sqrt{q^h}$ would give a negative number of hydrogen filling stations, see equation 3.

is important. If this cost is too high, other carbon abatement options than converting to hydrogen cars should be considered.

In case $\Theta < 0$, there are three real roots. Two of the roots are always positive (see appendix B), and represents potential market equilibriums. Of these two possible Nash equilibriums; one will be with high sales and one will be with low sales of hydrogen cars. The third root is always negative (see appendix B), and thus not feasible. On the other hand, the corner solution $q^h = 0$ is still a Nash equilibrium.

The variable Θ is less than zero when both $a < 0$ and $b <\mid 2\sqrt{\frac{a^3}{27}} \mid$. This gives the following proposition:

Proposition 2 *In case the additional cost of hydrogen cars and the additional cost of hydrogen supply are not to high, and similarly, the entry cost of hydrogen filling stations is also not too high, such that* $\frac{25tf^h}{64} < \frac{\left(m\bar{\Gamma}-\Delta\omega-\Delta c+\tau+\frac{5}{4}\sqrt{t}\right)^3}{27\bar{\Gamma}}$, *there are three Nash equilibriums in the market game for personal transport.*

We denote these three equilibriums as q^{ho}, q^{h*} and q^{h**}, zero, low and high equilibriums of hydrogen cars, respectively. As shown in the appendix the two positive equilibriums do not coincide. In fact, for each set of parameters it is straight forward to calculate one specific q^{h*} and one specific q^{h**}.

6 The existence of lock-in

Central to the discussion about lock-in is the environmental costs of traditional cars. We assume that hydrogen cars are emission free. Further, since the life time mileage per car is given, total emission from traditional cars can be written: $e(1 - q^h)$ where e, as mentioned, is a proportional factor linking emissions and life time mileage per car. Finally, we assume a constant marginal damage cost which is equal to the international quota price on carbon emissions σ.

There is no producer surplus in the car industry, that is, prices equal the constant marginal costs for each type of car. Furthermore, producer surpluses in the hydrogen filling station sector are zero due to the free entry condition. In the gasoline filling station sector ex post profits may turn negative i.e. given that hydrogen filling stations enter and $q^g < 1$, all gasoline station owners will regret their decision to enter. However, as long as the cost of the station is sunk, the best they can do is to continue selling gasoline.

Welfare in any equilibrium, $q^g = 1 - q^h$, is then given:

$$W(q^g, q^h) = \int_{l}^{m-q^h} [\lambda_x \Gamma^g - \omega^g - p^g - \delta^g] d\lambda + \int_{m-q^h}^{m} [\lambda_x \Gamma^h - \omega^h - p^h - \delta^h] d\lambda \quad (10)$$

$$- [p^g - c^g - \tau]\, q^h + (\tau - \sigma e)(1 - q^h),$$

where $m - q^h$ is equal to λ_x^* i.e. the marginal consumer. Further, the first term is the consumer surplus of those buying a traditional car, and the second term is the consumer surplus of those buying a hydrogen car. The third term is the loss in profit of the gasoline station owners, that is, their mark-up times the loss in sales. The last term is net environmental costs i.e. the emission tax income subtracted the damage costs. Note that they cancel out since $\tau = \sigma e$.

After some rearranging (10) can be written:

$$W(q^g, q^h) = \frac{\bar{\Gamma}}{2}(q^h)^2 - \sqrt{t} q^h + \Gamma^g(m - \frac{1}{2}) - \omega^g - p^g - \delta^g \qquad (11)$$

The terms $\Gamma^g(m - \frac{1}{2}) - \omega^g - p^g - \delta^g$ constitutes total welfare if all consumers have traditional cars. Thus, the terms $\frac{\bar{\Gamma}}{2}(q^h)^2 - \sqrt{t} q^h$ can be interpreted as the net gain welfare from those that choose hydrogen cars.

Assume more equilibriums exist. We will first consider the case in which we observe no or low hydrogen car sales. The change in welfare ΔW going from the low to the high hydrogen car sales equilibrium can then be written:

$$\Delta W = \left[\frac{(q^{h*} + q^{h**})}{2} \bar{\Gamma} - \sqrt{t} \right] \Delta q^h,$$

where $\Delta q^h = q^{h**} - q^{h*}$. The term$\sqrt{t}$ is the marginal loss in producer surplus, while the term $\frac{(q^{h*}+q^{h**})}{2}\bar{\Gamma}$ can be interpreted as the marginal gain in consumer surplus. The result is summarized in the following proposition:

Proposition 3 *We have a lock-in situation if and only if the marginal gain in consumer surplus is larger than the marginal loss in producer surplus i.e.* $\frac{(q^{h**}+q^{h*})}{2}\bar{\Gamma} > \sqrt{t}$.

The loss of the gasoline station owners has much in common with the creative destruction described by Aghion and Howitt (1998). As noted by Aghion and Howitt we may have too much R&D in the laissez-faire economy if the business stealing effect dominates, which is possible if the size of the innovations are small. This is precisely the case in our model whenever $\bar{\Gamma}$ is small such that $\frac{(q^{h**}+q^{h*})}{2}\bar{\Gamma} < \sqrt{t}$.

The following corollary to Proposition 3 states the policy rule:

Corollary 4 *If governments disregard losses in producer surplus, governments should always intervene when the market gives a signal (i.e. some use of hydrogen cars). On the other hand, if governments pay equal attention to consumer and producer surplus, there is a danger of excess momentum even if the market gives a signal.*

Consumers with a high preference for car quality may buy hydrogen cars, even with the small network in filling stations they can expect. Then, there is an amount of other consumers that would have chosen hydrogen cars, if the network had been larger. Without intervention the market stays in the low Nash equilibrium. However, with policy induced incentives the market can be coordinated to the high equilibrium. These incentives could for instance be hydrogen filling station entry cost subsidies, or simply, hydrogen supply requirements for some proportion of the filling stations. However, the decision to intervene clearly requires a lot of information gathering even if some use of hydrogen cars takes place already.

Secondly, consider the case of *excess momentum*. That is, we observe high sales of hydrogen cars, but the high hydrogen car sales equilibrium represents a welfare loss. In this case the policy makers should *not* seek to coordinate the market to one of the lower hydrogen car sales equilibriums. This is because a move from a high to a low hydrogen car sales equilibrium implies a loss for hydrogen station owners that exceeds the gain for the gasoline station owners. The loss for hydrogen station owners is $[p^h - c^h](q^{h*} - q^{h**})$, which can be written as $\sqrt{\frac{tf^h}{q^{h**}}}(-\Delta q^h)$. The change in welfare ΔW^h when going from a high hydrogen car sales to a low hydrogen car sales equilibrium can then be written:

$$\Delta W^h = \left[\frac{\bar{\Gamma}}{2}(q^{h**} + q^{h*}) - \sqrt{t} + \sqrt{\frac{tf^h}{q^{h**}}}\right](-\Delta q^h),$$

which implies the following proposition:

Proposition 5 *When there is excess momentum in the market for personal transport policy makers should not seek to coordinate the market to one of the lower hydrogen car sales equilibriums.*

Proof. $f^h \geq 1$ and $q^{h**} < 1$ gives that $\sqrt{t} < \sqrt{\frac{tf^h}{q^{h**}}}$. Since $\frac{\bar{\Gamma}}{2}(q^{h**}+q^{h*}) > 0$ and $-\Delta q^h < 0$, it then follows that $\Delta W^h < 0$. ■

Transition and adjustment costs are not included in our static one stage model. It may of course be that these costs are so large that the *status quo* is welfare superior to the high hydrogen equilibrium even though Proposition 3 is fulfilled. This point is not further developed here.

7 Discussion and conclusion

Since we have been looking at a small country, we have not incorporated potential effects from learning by doing in our model. Learning effects with high spillovers between firms could alter our results as the hydrogen technology would improve through utilization. Learning is often used as an argument for government intervention in the market for hydrogen based transport, see for instance Farrell, Keith and Corbett (2003). However, we believe that the greatest technological challenges for widespread use of hydrogen cars, hydrogen storage and fuel cells, could develop in niche markets like markets for cell phones and laptops. Hence, active support to the hydrogen car to generate technological learning is likely unnecessary. Of course, further research is needed to analyze this properly.

Moreover, we do not study stability of Nash equilibriums or dynamics in the present paper. This seems to be a fruitful future venue for research, as a key issue for the lock-in problem is how the economy moves from one equilibrium to another.

In this paper our research question has been whether governments should endorse the introduction of the hydrogen car or let the market settle the matter on its own. If the costs of hydrogen car and filling station technology are large, there are neither consumers with hydrogen cars nor filling stations offering hydrogen fuel. Due to these high costs there is no lock-in in the market, even though potential network externalities are present.

Most technology experts seem to agree that the hydrogen transport technology is not yet fully developed. For instance, fuel cells needed to convert hydrogen into energy are still expensive and not sufficiently reliable, and the problem of storing hydrogen in the car in a convenient way is also only partially solved (OECD, 2004, Service, 2004 and Wald, 2004). Hence, given the present state of the hydrogen technology, endorsement of the hydrogen car with infrastructure subsidies or subsidies to hydrogen car buyers do not look especially desirable. Rather, it seem to us, that governments should focus on R&D in order to improve the most serious technological weaknesses (if that at all is possible).

When quality improves and costs become low enough, there may be some consumers with high preferences for car quality that adopt hydrogen cars, and some hydrogen filling stations that get established. In such a case there will be a number of other consumers that would have chosen the hydrogen car if the network of filling stations were larger. This may be a lock-in situation, but only if the increase in consumer surplus outweights the loss from unused gasoline station capacity.

Our analysis points to the fact that this is indeed difficult to determine. To make the right decision about intervention policy makers need substantial information on three key issues; the maturity of the hydrogen technology, the consumers benefit of making a change in technology, and the loss from unused capacity in the market. Hence, we are tempted to conclude that governments should let the market settle the matter, and only ensure implementation of the correct carbon taxes on fossil fuels.

References

[1] Aghion P. and Howitt P. (1998), "Endogenous Growth Theory". The MIT Press.

[2] Andreoni J. (1990), "Impure Altruism and Donations to the Public Good: A theory of warm glow giving". The Economic Journal 100, p. 464-477.

[3] Caldeira et al. (2002), "Advanced Technology Paths to Global Climate Stability: Energy for the Greenhouse Planet", Science 298.

[4] Cho A. (2004), "Fire and ICE: Revving up for H_2". Science 305/5686, p. 964-965.

[5] Conrad K. (2005),"Network effects, Compatibility and the Environment: The Case of Hydrogen Powered Cars". Paper presented at European Conference of Environmental and Resource Economists, Bremen 2005.

[6] Farrell A. E., D. W. Keith and J. J. Corbett (2003), "A strategy for introducing hydrogen into transportation". Energy Policy 31. p. 1357-1367.

[7] Farrell J. and G. Saloner (1985), "Standardization, compatibility, and innovation". Rand Journal of Economics 16, p. 70-83.

[8] Farrell J. and G. Saloner (1986), "Installed Base and Compatibility: Innovation, Product Preannouncements, and Predation". American Economic Review 76, p. 940-955.

[9] Katz M. L. and C. Shapiro (1986), "Technology Adoption in the Presence of Network Externalities", Journal of Political Economy 94, p. 822-841.

[10] Melaina M. W. (2003), "Initiating hydrogen infrastructures: pre-

liminary analysis of a sufficient number of initial hydrogen stations in the US". Hydrogen Energy 28, p. 743-755.
[11] OECD (2004), "Hydrogen and Fuel Cells", Review of National R&D Programs, OECD, Paris.
[12] Padro C. E. and V. Putsche (1999), "Survey of the Economics of Hydrogen Technologies", Report NREL/TP-570-27079, National Renewable Energy Laboratory.
[13] Salop S. (1979), "Monopolistic Competition with Outside Goods", Bell Journal of Economics 10, p.141 - 156.
[14] Service R. F. (2004), "The hydrogen backclash". Science 305/5686, p. 958-963.
[15] Shaked A. and J. Sutton (1982), "Relaxing Price Competition Through Product Differentiation", Review of Economic Studies XLIX, p. 3-13.
[16] Spence M.A. (1981), The Learning curve and competition, *The Bell Journal of Economics*, 12, p. 49-70.
[17] Tirole J. (1997), "The Theory of Industrial Organization". The MIT Press.
[18] Turnbull H.W. (1939), "Theory of Equations". Oliver and Boyd. Edinburgh.
[19] Unruh G. C. (2000), "Understanding carbon lock-in". Energy Policy 28, p. 817-830.
[20] Unruh G. C. (2002), "Escaping carbon lock-in". Energy Policy 30, p. 317-325.
[21] Wald M. L. (2004), "Questions about a hydrogen economy". Scientific American 290/5, p. 40-48.

8 Appendix A - Horizontal differentiation

Here we follow the original model of Hotelling (see e.g. Tirole, 1997, page 279). Zero on the line represents traditional technology, and 1 represents hydrogen technology. The closer a consumer is either to zero or one, the stronger preference she has to the respective technology. The utility of consumer x from buying a traditional car is then:

$$U_x^g = \Gamma - \omega^g - p^g - \delta^g - r\theta_x,$$

where Γ is the gross utility of buying a car, the parameter $\theta_x \in [0, 1]$ is the location of the consumer on the line, and r is a preference parameter.

Respectively, the utility of consumer x from buying a hydrogen car is:

$$U_x^h = \Gamma - \omega^h - p^h - \delta^h - r(1 - \theta_x).$$

The marginal consumer being indifferent between a hydrogen car and a traditional car is then given by:

$$\theta_x^* = \frac{\omega^h + p^h + \delta^h - \omega^g - p^g - \delta^g + r}{2r}.$$

The demand D facing the two types of cars can then be written:

$$q^i = \begin{cases} \frac{\omega^h + p^h + \delta^h - \omega^g - p^g - \delta^g + r}{2r} \text{ for traditional cars} \\ \frac{r - \omega^h - p^h - \delta^h + \omega^g + p^g + \delta^g}{2r} \text{ for hydrogen cars} \end{cases}, \; i = g, h.$$

By inserting (2), (3) and (4) into the demand function for traditional cars and hydrogen cars we obtain the market shares of consumers for the two technologies. For traditional cars we have:

$$q^g = \frac{r + \Delta\omega + \Delta c - \tau + \frac{5\sqrt{t}}{4}\left[\frac{\sqrt{f^h}}{\sqrt{q^h}} - 1\right]}{2r}, \tag{12}$$

and for hydrogen cars we get:

$$q^h = \frac{r - \Delta\omega - \Delta c + \tau - \frac{5\sqrt{t}}{4}\left[\frac{\sqrt{f^h}}{\sqrt{q^h}} - 1\right]}{2r}. \tag{13}$$

Rearranging equation (13), setting $z = \sqrt{q^h}$, we get:

$$z^3 - \frac{\left[r + \tau - \Delta\omega - \Delta c + \frac{5\sqrt{t}}{4}\right]}{2r} z + \frac{5\sqrt{tf^h}}{8r} = 0. \tag{14}$$

Basically equation (9) and (14) have the same properties. Using (14) instead of (9) gives the same results with respect to the existence of lock-in, though with a different interpretation of consumers' preferences.

Welfare in the horizontal differentiation case is given:

$$W^h(q^h) = \int_0^{1-q^h} [\Gamma - \omega^g - p^g - \delta^g - r\theta] d\theta \tag{15}$$

$$+ \int_{1-q^h}^{1} [\Gamma - \omega^h - p^h - \delta^h - r(1-\theta)] d\theta - [p^g - c^g - \tau] q^h + (\tau - \delta e)(1 - q^h).$$

Which can be solved:

$$W^h(q^h) = \left[(\Gamma - \omega^g - p^g - \delta^g)\, \theta - \frac{r}{2}\theta^2 \right]_0^{1-q^h}$$

$$+ \left[\left(\Gamma - \omega^h - p^h - \delta^h \right) \theta + \frac{r}{2}(1-\theta)^2 \right]_{1-q^h}^{1} - \sqrt{t}q^h$$

$$\Longleftrightarrow$$

$$W^h(q^h) = \Gamma - \omega^g - p^g - \delta^g - \frac{r}{2} + \left(r - \omega^h - p^h - \delta^h + \omega^g + p^g + \delta^g \right) q^h - r(q^h)^2 - \sqrt{t}q^h$$

where we have used that $\sqrt{t} = p^g - c^g - \tau$ and $\tau = \delta e$.

Using that $q^h = \frac{r - \omega^h - p^h - \delta^h + \omega^g + p^g + \delta^g}{2r}$ this then yields:

$$W^h(q^h) = r(q^h)^2 - \sqrt{t}q^h + \Gamma - \omega^g - p^g - \delta^g - \frac{r}{2},$$

which basically is the same as (11).

9 Appendix B - The cubic equation

The cubic equation:

$$z^3 + az + b = 0 \tag{16}$$

have three solutions that can be both real and complex numbers (Turnbull, 1939). This is determined by the sign of Θ, where $\Theta = \frac{b^2}{4} + \frac{a^3}{27}$. When $\Theta > 0$, there are two complex roots and one real root to the equation. When $\Theta < 0$, all three roots are real. We disregard the case $\Theta = 0$.

Complex solutions together with negative solutions are outside the feasible set of z. This is because $\sqrt{q^h} = z$, where $q^h \in [0, 1]$.

When $\Theta > 0$, we can use the formula of Cardan

$$z = \sqrt[3]{-\frac{1}{2}b + \sqrt{\Theta}} + \sqrt[3]{-\frac{1}{2}b - \sqrt{\Theta}}$$

directly to find the real root. This root can be either positive or negative in a general case. In our model we have a positive b, thus the root is always negative since $\mid \sqrt[3]{-\frac{1}{2}b + \sqrt{\Theta}} \mid < \mid \sqrt[3]{-\frac{1}{2}b - \sqrt{\Theta}} \mid$ when $b > 0$.

When $\Theta < 0$, we can use the trigonometric solution of the cubic equation

$$z = 2\sqrt{\frac{-a^3}{27}}\cos(\frac{\theta + 2k\pi}{3}) \qquad (17)$$

where θ is given by

$$\cos\theta = \frac{-b}{2\sqrt{\frac{-a^3}{27}}} \qquad (18)$$

and $k = 1$,2 or 3 represents the three different real roots. The trigonometric solution gives the multiple equilibria in our model. In order to do welfare analysis we need to determine which k gives a feasible solution, and which k gives the higher realization of q^h.

A1.1.- Feasible realizations when $\Delta < 0$ A negative realization of z is not feasible. The sign of z is determined by sign of $\cos(\frac{\theta+2k\pi}{3})$ since $\sqrt{\frac{-a^3}{27}} > 0$ when $\Theta < 0$. To determine this sign we first look into the range of θ. Note that since $\Theta = \frac{b^2}{4} + \frac{a^3}{27} < 0$ we have that $\mid b \mid < 2\sqrt{\frac{-a^3}{27}}$. Since $b > 0$, this is equal to $\frac{-b}{\sqrt{\frac{-a^3}{27}}} > -1$ with $\frac{-b}{\sqrt{\frac{-a^3}{27}}} \in (-1, 0)$. This gives the range for $\cos\theta$ from equation (cos-equ), $\cos\theta \in (-1, 0)$. We calculate the values for θ,

$$\theta = \arccos\{\frac{-b}{\sqrt{\frac{-a^3}{27}}} \in (-1, 0)\}$$

which gives $\theta \in (\frac{\pi}{2}, \pi)$.

We have that the sign of z is determined by the sign of $\cos(\frac{\theta+2k\pi}{3})$, and that $\theta \in (\frac{\pi}{2}, \pi)$. For the lower boundary of θ the value of $\cos(\frac{\theta+2k\pi}{3})$ is given by $\cos(\frac{\pi}{6} + \frac{2k\pi}{3} + \epsilon)$, where ϵ is a number close to zero. This gives the sign of $\cos(\frac{\theta+2k\pi}{3})$ from the three different roots represented by $k = 1$,2 or 3

k	$\cos(\frac{\pi}{6} + \frac{2k\pi}{3} + \epsilon)$
0 :	$\cos(\frac{\pi}{6} + \epsilon) > 0$
1 :	$\cos(\frac{5\pi}{6} + \epsilon) < 0$
2 :	$\cos(\frac{3\pi}{2} + \epsilon) > 0$

For the higher boundary of θ the value of $\cos(\frac{\theta+2k\pi}{3})$ is given by $\cos(\frac{\pi}{3} + \frac{2k\pi}{3} - \epsilon)$. This gives the sign of $\cos(\frac{\theta+2k\pi}{3})$ from the three different roots represented by $k = 1$,2 or 3

k	$\cos(\frac{\pi}{3} + \frac{2k\pi}{3} - \epsilon)$
$0:$	$\cos(\frac{\pi}{3} - \epsilon) > 0$
$1:$	$\cos(\pi - \epsilon) < 0$
$2:$	$\cos(\frac{5\pi}{3} - \epsilon) > 0$

This gives, by inspection of the cosine curve, that there will always be one negative root and two positive roots to z when $\Theta < 0$. The positive roots are for $k = 0$ and $k = 2$, the negative root is for $k = 1$.

The roots solve the value of z, which again gives the value of q^h. A negative solution of z is not feasible since $\sqrt{q^h} = z$, where $q^h \in [0, 1]$. This gives the following result: *In the case $\Theta < 0$ there are only two feasible realizations of q^h, for $k = 0$ and $k = 2$.*

The root of z for $k = 0$ and the root of z for $k = 2$ do not coincide. This means that there are two different feasible realizations of q^h.

A1.2.- The greater root from $k = 0$ and $k = 2$ Which k gives the greater solution of z depends on the value of $\cos(\frac{\theta+2k\pi}{3})$ in the possible range of $\theta \in (\frac{\pi}{2}, \pi)$. The following table gives the range of $\cos(\frac{\theta+2k\pi}{3})$ in the range of θ for $k = 0$ and $k = 2$:

k	θ	$\cos(\frac{\theta+2k\pi}{3})$
$0:$	$\theta \in (\frac{\pi}{2}, \pi)$	$\cos(\frac{\theta}{3}) \in (\cos\frac{\pi}{6}, \cos\frac{\pi}{3})$
$2:$	$\theta \in (\frac{\pi}{2}, \pi)$	$\cos(\frac{\theta+4\pi}{3}) \in (\cos\frac{3\pi}{2}, \cos\frac{5\pi}{3})$

This gives that $\cos(\frac{\theta+2\pi}{3})$ is greater when $k = 0$ than $k = 2$ in the range of θ.

Proof. First define $\mu \equiv \cos(\frac{\theta+2\pi}{3})$. When $k = 0$, then μ is decreasing in the range $\theta \in (\frac{\pi}{2}, \pi)$. When $k = 2$, then μ is increasing in the range $\theta \in (\frac{\pi}{2}, \pi)$. We also have that when $\theta = \pi$, then μ is equal for $k = 0$ and $k = 2$. ■

This gives the following result: *In the case $\Theta < 0$ there are two realizations of q^h, where the high realization is for $k = 0$ and the low realization is for $k = 2$.*