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# Compensated Variation in Random Utility Models

**Abstract:**
In this paper we introduce the notion of random expenditure function and derive the distribution of the expenditure function and corresponding compensated choice probabilities in the general case when the (random) utilities are nonlinear in income. We also derive formulae for expenditure and choice under price (policy) changes conditional on the initial utility level. This is of particular interest for welfare measurement because it enables the researcher to analyze the distribution of Compensating variation.

**Keywords:** Random expenditure function, Compensated choice probabilities, Compensating variation.

**JEL classification:** C25, D61

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# 1. Introduction

In this paper we introduce the notion of the expenditure function and compensated demand within the theory of discrete choice. Since the theory of discrete choice is based on a random utility formulation, it follows that the corresponding expenditure function is random. In the context of measuring the welfare effect of changing prices or non-pecuniary attributes it is useful to apply the expenditure function to derive Equivalent Variation and Compensating Variation (cv). When the (random) utility function is nonlinear in income analytic formulae for the distribution of cv has so far not been available. See Herriges and Kling (1999) for a review of the state of the art as well as previous contributions. McFadden (1995) has developed a Monte Carlo simulator for computing cv in random utility models and Herriges and Kling (1999) have investigated the empirical consequences of nonlinear income effects based on a particular empirical application.[1] Aaberge et al. (1995) have used a Monte Carlo simulation to compute equivalent variation.

By using the stochastic structure of random expenditure function as a point of departure we demonstrate in this paper that one can obtain explicit analytic formulae for the distribution of cv for general random utility models. In the case where the model belongs to the Generalized Extreme Value class the formulae become particularly simple.

Let us briefly review the definition of cv in random utility models with particular focus on the challenge of calculating the distribution of cv. Let $U_j$ denote the utility of alternative j and assume that

$$U_j = v_j\left(w_j, y\right) + \varepsilon_j,$$

where y represents income, $w_j$ is price or a vector of attributes including price associated with alternative j, $v_j(\cdot),\ j=1,2,...,$ are deterministic functions and $\varepsilon_j,\ j=1,2,...,$ are random terms with joint distribution that does not depend on the structural terms $\left\{v_j(\cdot)\right\}$.[2] Then (ignoring the choice set in the notation) cv is defined implicitly through

$$\max_j\left(v_j\left(w_j^0, y^0\right) + \varepsilon_j\right) = \max_j\left(v_j\left(w_j^1, y^0 + cv\right) + \varepsilon_j\right)$$

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[1] I recently became aware of the work by Karlström (1999), who has independently obtained many of the same results as in this paper, although the proofs are different.

[2] Note that the present notation accommodates the specification $v_j\left(w_j, y\right) = \tilde{v}\left(w_{1j}, y - w_{2j}\right)$ where the function $\tilde{v}$ does not depend on j and $w_{1j}$ represents non-pecuniary attributes and $w_{2j}$ the price (or user cost) associated with alternative j.

where $\{w_j^0, y^0\}$ represent initial attributes and income, and $\{w_j^1\}$ are the attributes implied by the policy. Here it is assumed that the random terms $\{\varepsilon_j\}$ are not affected by the policy intervention. Clearly, cv becomes a random variable that depends on all the error terms and all the attributes and initial income. From an analytic point of view the difficulty of attaining a formulae for the distribution of cv stems from the fact that when the new attributes $\{w_j^1\}$ are introduced then the alternative that yields maximum utility may be different from the one that maximized utility initially. In other words, the individual agent may switch from the alternative chosen initially to a new one, when the policy is introduced. If, however, the random terms initially and after the intervention are stochastically independent, then the complexity of the problem reduces drastically.

The paper is organized as follows. In Section 2 the discrete choice framework is presented, and in Section 3 compensating choice probabilities and the random expenditure function are defined and the corresponding distribution functions are derived. In Section 4 we derive compensated choice probabilities and the distribution of the expenditure function under price changes conditional on a utility level equal to the initial level under different assumptions about the random terms of the utility function.

## 2. The setting

We consider a setting in which a consumer faces a set B of feasible alternatives (products), which is a subset of the universal set S of alternatives, $S = \{0,1,2,\ldots,M\}$, where 0 indexes the alternative "no purchase". The consumers utility function of alternative j is assumed to have the form

(1) $$U_j = v_j(w_j, y) + \varepsilon_j$$

where y denotes income and $w_j$ is a vector of attributes including price of alternative j. The function $v_j(\cdot)$ is assumed to be continuous, decreasing in the first argument and strictly increasing in the second, and may depend on j. For notational simplicity non-pecuniary attributes are suppressed in the notation.

Let $F^B(\cdot)$ denote the joint cumulative distribution function of $\{\varepsilon_k, k \in B\}$. We assume that $F(\cdot) \equiv F^S(\cdot)$ possesses a density. Thus the probability of ties is zero. Then it is well known that one can express the Marshallian choice probabilities by a simple formulae. For notational simplicity, let $B = \{0,1,2,...,m\}$. Then the *Marshallian* choice probabilities are given by

(2) $$P_j(B, y, \mathbf{w}) \equiv P\left(U_j = \max_{k \in B} U_k\right) = \int F_j^B\left(u - v_0(w_0, y), u - v_1(w_1, y), ..., u - v_m(w_m, y)\right) du$$

where $F_j^B(x_1, x_2, ..., x_m)$ denotes the partial derivative with respect to $x_j$ and $\mathbf{w} = (w_1, w_2, ..., w_M)$. Here it is understood that income and prices, (y,**w**) are *given*.

If $\{\varepsilon_j\}$ are random variables with multivariate extreme value distribution, then $G^B$ defined by

(3) $$\exp\left(-G^B(x_0, x_1, ..., x_M)\right) \equiv P\left(\bigcap_{k\in B} (\varepsilon_k \le x_k) \middle| y, \mathbf{w}\right)$$

has the property[3]

(4) $$G^S(x_0, x_1, ..., x_M) = e^{-z} G^S(x_0 - z, x_1 - z, ..., x_M - z)$$

for $z \in R$. The corresponding Marshallian choice probability is given by

(5) $$P_j(B, y, \mathbf{w}) = -\frac{G_j^B\left(-v_0(w_0, y), -v_1(w_1, y), ..., -v_m(w_m, y)\right)}{G^B\left(-v_0(w_0, y), -v_1(w_1, y), ..., -v_m(w_m, y)\right)}.$$

This formulae is well known and is found in a complete analogous form in for example McFadden (1981).

# 3. The random expenditure function and compensated (Hicksian) choice probabilities

We now proceed to discuss the notion of expenditure function that corresponds to the above setting.

Let $Y_B(\mathbf{w},u)$ be the expenditure function defined as

(6) $$Y_B(\mathbf{w}, u) = \left\{z : \max_{k\in B}\left(v_k(w_k, z) + \varepsilon_k\right) = u\right\}.$$

The expenditure function can be readily computed as follows. Let $Y(w_k, u, k)$ be determined by

$$v_k\left(w_k, Y(w_k, u, k)\right) = u - \varepsilon_k.$$

Due to the fact $v_k(w_k, y)$ is strictly increasing in y, $Y(w_k, u, k)$ is uniquely determined. We realize that the expenditure function equals

---

[3] In addition G must satisfy a number of regularity conditions to ensure that $\exp(-G(x_0,x_1,...,x_M))$ is a proper distribution function, cf. McFadden (1978).

$$Y_B(\mathbf{w},u) = \min_{k\in B} Y(w_k,u,k).$$

With probabililty one the set Y(**w**,u) is a singleton because for $z \neq z^*$, we have

$$P\left(\max_{k\in B}\left(v_k(w_k,z)+\varepsilon_k\right) = \max_{k\in B}\left(v_k\left(w_k,z^*\right)+\varepsilon_k\right)\right) = 0.$$

We thus obtain the next result.

**Theorem 1**

*Let* $B=\{0,1,2,\ldots,m\}$. *The distribution of the expenditure function is given by*

$$P\left(Y_B(\mathbf{w},u)\leq y\right) = 1-F^B\left(u-v_0\left(w_0,y\right),u-v_1(w_1,y),\ldots,u-v_m\left(w_m,y\right)\right)$$

*where* $F^B(\cdot)$ *denotes the joint distribution of* $\{\varepsilon_k, k\in B\}$.

A proof of Theorem 1 is given in the appendix.

As mentioned above, the notion of Hicksian- or compensated choice probabilities does not seem to have appeared previously in the literature. Below we propose a natural definition.

**Definition 1**

*By Hicksian choice probabilities,* $\left\{P_j^h(B,u,\mathbf{w})\right\}$, *we mean*

$$P_j^h(B,u,\mathbf{w}) \equiv P\left(v_j\left(w_j,Y_B(\mathbf{w},u)\right)+\varepsilon_j = \max_{k\in B}\left(v_k\left(w_k,Y_B(\mathbf{w},u)\right)+\varepsilon_k\right)\right).$$

The interpretation of $P_j^h(B,u,\mathbf{w})$ is as the probability of choosing $j\in B$ given that the utility level is given and equal to u. For example, if prices change the consumers are given income compensation so as to maintain a given utility level.

**Theorem 2**

*The Hicksian choice probabilities can be expressed as*

$$P_j^h(B,u,\boldsymbol{w}) = \int_0^\infty \frac{F_j^B\left(u-\mathrm{v}_0(w_0,y),u-\mathrm{v}_1(w_1,y),\dots,u-\mathrm{v}_m(w_m,y)\right)\sum_{k\in B}F_k^B\left(u-\mathrm{v}_0(w_0,y),u-\mathrm{v}_1(w_1,y),\dots,u-\mathrm{v}_m(w_m,y)\right)\mathrm{v}_k(w_k,dy)}{\sum_{k\in B}F_k^B\left(u-\mathrm{v}_0(w_0,y),u-\mathrm{v}_1(w_1,y),\dots,u-\mathrm{v}_m(w_m,y)\right)}.$$

A proof of Theorem 2 is given in the appendix.

From Theorem 2 we realize that one can calculate the Hicksian choice probabilities readily provided the cumulative distribution $F^B(\cdot)$ is known since only a one dimensional integral is involved in the formulae for $P_j^h(B,u,\mathbf{w})$.

**Theorem 3**

*Suppose F(·) is a multivariate extreme value distribution. Then the Hicksian choice probabilities are given by*

$$P_j^h(B,u,\boldsymbol{w}) = \int_0^\infty P_j(B,y,\boldsymbol{w})P\left(Y_B(\boldsymbol{w},u)\in dy\right).$$

A proof of Theorem 3 is given in the appendix.

The reason for the simplification of the Hicksian choice probabilities expressed in Theorem 3 is that when the error terms in the utility function are multivariate extreme value distributed then the indirect utility is stochastically independent of which alternative that is chosen.

# 4. The probability distribution of the expenditure function and the choice under price changes conditional on the initial utility level

We shall next consider the problem of characterizing the Hicksian choice probabilities when the utility level equals the (indirect) utility under prices and income that differ from the current prices and incomes. To this end we consider a two period setting. In period one (the initial period) the prices and income are $\left(\mathbf{w}^0, y^0\right)$. In the second period (current period) the prices are **w**. As above, it is assumed that the respective random terms remain unchanged under price changes. In general, when prices (or other attributes) change it may yield a decrease or an increase in the agent's indirect utility. However, the highest utility may no longer be attained at the alternative chosen initially, and consequently the agent will switch to a new alternative, namely the one that maximizes utility under the new price regime. In the current setting, however, the (indirect) utility level is kept fixed and equal to the initial level. But the agent may still switch from the initially chosen alternative to a new one because, after the price changes, the utility of the initially chosen alternatives may no longer coincide with the new indirect utility.

Let us first consider the distribution of $Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right)$, where

(7) $$V\left(\mathbf{w}^0, y^0\right) = \max_{k \in B}\left(v_k\left(w_k^0, y^0\right) + \varepsilon_k\right).$$

The interpretation of $Y_B\left(\mathbf{w}, V\left(y^0, \mathbf{w}^0\right)\right)$ is as the expenditure function conditional on the utility level that corresponds to income level $y^0$. Hence, the corresponding Compensating variation measure equals $y_0 - Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right)$.

Let $J_B\left(\mathbf{w}^0, y^0\right)$ denote the initial choice from B and let $J_B^*\left(\mathbf{w}^0, y^0, \mathbf{w}\right)$ denote the current choice from the choice set B, given the current and initial prices and income $\left(\mathbf{w}, \mathbf{w}^0, y^0\right)$, and given that the current utility level equals the initial one, $V\left(\mathbf{w}^0, y^0\right)$.

**Theorem 4**

*Let*

$$h_j\left(w_j^0, y^0, w_j, y\right) = max\left(v_j\left(w_j, y\right), v_j\left(w_j^0, y^0\right)\right).$$

*Then*

$$P\left(Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right) \in dy, J_B\left(\mathbf{w}^0, y^0\right) = i, J_B^*\left(\mathbf{w}^0, y^0, \mathbf{w}\right) = j\right)$$
$$= v_j\left(w_j, dy\right) \int F_{ij}^B\left(u - h_0\left(w_0^0, y^0, w_0, y\right), u - h_1\left(w_1^0, y^0, w_1, y\right), \ldots, u - h_m\left(w_m^0, y^0, w_m, y\right)\right) du$$

*when* $i \neq j$, $i, j \in B$ *and* $y_{jj}\left(w_j^0, y^0, w_j\right) \leq y < y_{ii}\left(w_i^0, y^0, w_i\right)$, *where* $y_{ii}\left(w_i^0, y^0, w_i\right)$ *is determined by* $v_i\left(w_i^0, y^0\right) = v_i\left(w_i, y_{ii}\left(w_i^0, y^0, w_i\right)\right)$. *Otherwise*

$$P\left(Y_B\left(\boldsymbol{w},V\left(\boldsymbol{w}^0,y^0\right)\right)\in dy, J_B\left(\boldsymbol{w}^0,y^0\right)=i, J_B^*\left(\boldsymbol{w}^0,y^0,\boldsymbol{w}\right)=j\right)=0 .$$

*For* $j=i\in B$,

$$P\left(Y_B\left(\boldsymbol{w},V\left(\boldsymbol{w}^0,y^0\right)\right)=y, J_B\left(\boldsymbol{w}^0,y^0\right)=J_B^*\left(\boldsymbol{w}^0,y^0,\boldsymbol{w}\right)=i\right)=P_i\left(B,y^0,\boldsymbol{w}^0\right)$$

*for* $y=y_{ii}\left(w_i^0,y^0,w_i\right)$, *and*

$$P\left(Y_B\left(\boldsymbol{w},V\left(\boldsymbol{w}^0,y^0\right)\right)\in dy, J_B\left(\boldsymbol{w}^0,y^0\right)=J_B^*\left(\boldsymbol{w}^0,y^0,\boldsymbol{w}\right)=i\right)=0$$

*for* $y\neq y_{ii}\left(w_i^0,y^0,w_i\right)$.

A proof of Theorem 4 is given in the appendix.

The result of Theorem 4 shows that only a one-dimensional integral is needed to calculate the joint probability density of

$$\left(\mathrm{Y_B}\left(\mathbf{w},\mathrm{V}\left(\mathbf{w}^0,\mathrm{y}^0\right)\right),\mathrm{J_B}\left(\mathbf{w}^0,\mathrm{y}^0\right),\mathrm{J_B^*}\left(\mathbf{w}^0,\mathrm{y}^0,\mathbf{w}\right)\right)$$

provided $\mathrm{F_{ij}}\left(\mathrm{x}_1,\mathrm{x}_2,...,\mathrm{x_m}\right)$ is known. However, in the multinomial Probit case where the utility function has normally distributed random components, a $\mathrm{m}-2$ dimensional integral is needed to calculate $\mathrm{F_{ij}}\left(\mathrm{x}_1,\mathrm{x}_2,...,\mathrm{x_m}\right)$.

The next result follows immediately from Theorem 4 by summing over $\mathrm{j}\in\mathrm{B}$ and integrating with respect to y.

**Corollary 2**

*The joint distribution of* $Y_B\left(\boldsymbol{w},V\left(\boldsymbol{w}^0,y\right)\right)$ *and* $J_B\left(\boldsymbol{w}^0,y^0\right)$ *is given by*

$$P\left(Y_B\left(\boldsymbol{w},V\left(\boldsymbol{w}^0,y^0\right)\right)>y,J_B\left(\boldsymbol{w}^0,y^0\right)=i\right)$$
$$=\int F_i^B\left(u-h_0\left(w_0^0,y^0,w_0,y\right),u-h_1\left(w_1^0,y^0,w_1,y\right),...,u-h_m\left(w_m^0,y^0,w_m,y\right)\right)du$$

*for* $i\in B$, *and* $0\leq y<y_{ii}\left(w_i^0,y^0,w_i\right)$.

The next corollary follows from Corollary 2 by summing over $i \in B$.

**Corollary 3**

*The c.d.f. of* $Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right)$ *is given by*

$$P\left(Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right) > y\right)$$
$$= \sum_{i \in B \setminus C\left(y^0, y, \mathbf{w}^0, \mathbf{w}\right)} \int F_i^B\left(u - h_0\left(w_0^0, y^0, w_0, y\right), u - h_1\left(w_1^0, y^0, w_1, y\right), \ldots, u - h_m\left(w_m^0, y^0, w_m, y\right)\right) du$$

*for* $y \geq 0$, *where*

$$C\left(y^0, y, \mathbf{w}^0, \mathbf{w}\right) = \left\{k : \mathrm{v}_k\left(w_k, y\right) \geq \mathrm{v}_k\left(w_k^0, y^0\right), k \in B\right\}.$$

The results obtained in Theorem 4 and Corollaries 2 and 3 are derived under the assumption that the choice set B is the same before and after the price change. However, these results also apply in cases where the choice set changes. Suppose for example that alternative 2 was available initially but is removed as part of a policy intervention. One can conveniently accommodate for this by letting $w_2$ become very large so that $v_2\left(w_2, y\right)$ becomes very small. As a result we obtain that

$$h_2\left(w_2^0, y, w_2, y\right) = v_2\left(w_2^0, y^0\right)$$

and that

$$2 \notin C\left(y^0, y, \mathbf{w}^0, \mathbf{w}\right).$$

From Corollary 3 it follows that the mean and variance of $Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right)$ can be calculated by the formulae

$$E\, Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right)$$

(8)
$$= \sum_{i \in B} \int_0^{y_{ii}\left(w_i^0, y^0, w_i\right)} \int F_i^B\left(u - h_0\left(w_0^0, y^0, w_0, y\right), u - h_1\left(w_1^0, y^0, w_1, y\right), \ldots, u - h_m\left(w_m^0, y^0, w_m, y\right)\right) du\, dy,$$

and

(9)

$$
\begin{aligned}
&\mathrm{E}\left(\mathrm{Y_B}\left(\mathbf{w},\mathrm{V}\left(\mathbf{w}^0,\mathrm{y}^0\right)\right)\right)^2\\
&=2\sum_{\mathrm{i}\in\mathrm{B}}\int_0^{\mathrm{y_{ii}}\left(\mathrm{w_i^0},\mathrm{y}^0,\mathrm{w_i}\right)}\mathrm{y}\int\mathrm{F_i^B}\left(\mathrm{u}-\mathrm{h_0}\left(\mathrm{w_0^0},\mathrm{y}^0,\mathrm{w_0},\mathrm{y}\right),\mathrm{u}-\mathrm{h_1}\left(\mathrm{w_1^0},\mathrm{y}^0,\mathrm{w_1},\mathrm{y}\right),\ldots,\mathrm{u}-\mathrm{h_m}\left(\mathrm{w_m^0},\mathrm{y}^0,\mathrm{w_m},\mathrm{y}\right)\right)\mathrm{du}\,\mathrm{dy}.
\end{aligned}
$$

We shall next discuss how the general results obtained above simplify in the case where $F(\cdot)$ is a multivariate extreme value distribution. This includes nested logit type models. For simplicity, we only state the joint density of

$$
\left(\mathrm{Y_B}\left(\mathbf{w},\mathrm{V}\left(\mathbf{w}^0,\mathrm{y}^0\right)\right),\mathrm{J_B}\left(\mathbf{w}^0,\mathrm{y}^0\right),\mathrm{J_B^*}\left(\mathbf{w}^0,\mathrm{y}^0,\mathbf{w}\right)\right)
$$

for the case when

$$
\mathrm{J_B}\left(\mathbf{w}^0,\mathrm{y}^0\right)\neq\mathrm{J_B^*}\left(\mathbf{w}^0,\mathrm{y}^0,\mathbf{w}\right).
$$

**Corollary 4**

*Suppose $F(\cdot)$ is a multivariate extreme value distribution. Then*

$$
\begin{aligned}
&P\left(Y_B\left(\boldsymbol{w},V\left(\boldsymbol{w}^0,y^0\right)\right)\in dy,J_B\left(\boldsymbol{w}^0,y^0\right)=i,J_B^*\left(\boldsymbol{w}^0,y^0,\boldsymbol{w}\right)=j\right)\\
&=\frac{\left(\tilde{G}_i^B\tilde{G}_j^B-\tilde{G}^B\tilde{G}_{ij}^B\right)h_j\left(w_j^0,y^0,w_m,dy\right)}{\tilde{G}^B}
\end{aligned}
$$

*when* $i\neq j, i,j\in B$ *and* $y_{jj}\left(w_j^0,y^0,w_j\right)\leq y<y_{ii}\left(w_i^0,y^0,w_i\right)$, *where*

$$
\tilde{G}^B=-logF^B\left(-h_0\left(w_0^0,y^0,w_0,y\right),-h_1\left(w_1^0,y^0,w_1,y\right),\ldots,-h_m\left(w_m^0,y^0,w_m,y\right)\right),
$$

$$
\tilde{G}_i^B=G_i^B\left(-h_0\left(w_0^0,y^0,w_0,y\right),-h_1\left(w_1^0,y^0,w_1,y\right),\ldots,-h_m\left(w_m^0,y^0,w_m,y\right)\right),
$$

$$
\tilde{G}_{ij}^B=G_{ij}^B\left(-h_0\left(w_0^0,y^0,w_0,y\right),-h_1\left(w_1^0,y^0,w_1,y\right),\ldots,-h_m\left(w_m^0,y^0,w_m,y\right)\right),
$$

*and $G^B(\cdot)$ is defined in (3).*

Thus, the result of Corollary 4 implies that in the case where the random terms are multivariate extreme value distributed, one can rather easily compute the joint density of the expenditure function and $J_B$ and $J_B^*$.

The next corollaries follow directly from Corollary 4.

**Corollary 5**

*Suppose F(·) is a multivariate extreme value distribution. Then we have, for* $i \in B$ *, and* $0 \le y < y_{ii}\left(w_i^0, y^0, w_i\right)$

$$P\left(Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right) > y, J_B\left(\mathbf{w}^0, y^0\right) = i\right)$$

$$= \frac{G_i^B\left(-h_0\left(w_0^0, y^0, w_0, y\right), -h_1\left(w_1^0, y^0, w_1, y\right), ..., -h_m\left(w_m^0, y^0, w_m, y\right)\right)}{G^B\left(-h_0\left(w_0^0, y^0, w_0, y\right), -h_1\left(w_1^0, y^0, w_1, y\right), ..., -h_m\left(w_m^0, y^0, w_m, y\right)\right)} .$$

**Corollary 6**

*Suppose F(·) is a multivariate extreme value distribution, then*

$$P\left(Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right) > y\right)$$

$$= \frac{\sum_{i \in B \setminus C\left(y^0, y, \mathbf{w}^0, \mathbf{w}\right)} G_i^B\left(-h_0\left(w_0^0, y^0, w_0, y\right), -h_1\left(w_1^0, y^0, w_1, y\right), ..., -h_m\left(w_m^0, y^0, w_m, y\right)\right)}{G^B\left(-h_0\left(w_0^0, y^0, w_0, y\right), -h_1\left(w_1^0, y^0, w_1, y\right), ..., -h_m\left(w_m^0, y^0, w_m, y\right)\right)}$$

*for* $y \ge 0$.

From eq. (8) and Corollary 6 we obtain that

(10)
$$\mathrm{EY}_B\left(\mathbf{w}, \mathrm{V}\left(\mathbf{w}^0, y^0\right)\right)$$

$$= \sum_{i \in B} \int_0^{y_{ii}\left(w_i^0, y^0, w_i\right)} \frac{G_i^B\left(-h_0\left(w_0^0, y^0, w_0, y\right), -h_1\left(w_1^0, y^0, w_1, y\right), ..., -h_m\left(w_m^0, y^0, w_m, y\right)\right) dy}{G^B\left(-h_0\left(w_0^0, y^0, w_0, y\right), -h_1\left(w_1^0, y^0, w_1, y\right), ..., -h_m\left(w_m^0, y^0, w_m, y\right)\right)}.$$

**Corollary 7**

*Suppose* $\varepsilon_0, \varepsilon_1, ..., \varepsilon_M$ *, are independent and extreme value distributed. Then for* $i \in B$ *, and* $0 \le y < y_{ii}\left(w_i^0, y^0, w_i\right)$

$$P\left(Y_B\left(\boldsymbol{w}, V\left(\boldsymbol{w}^0, y^0\right)\right) > y, J_B\left(\boldsymbol{w}^0, y^0\right) = i\right) = \frac{exp\left(\mathrm{v}_i\left(w_i^0, y^0\right)\right)}{\sum_{k \in B} exp\left(max\left(\mathrm{v}_k\left(w_k, y\right), \mathrm{v}_k\left(w_k^0, y^0\right)\right)\right)}.$$

From Corollary 7 we obtain the next result:

**Corollary 8**

*Suppose* $\varepsilon_0, \varepsilon_1, ..., \varepsilon_M$ *, are independent and extreme value distributed. Then*

$$P\left(Y_B\left(\boldsymbol{w}, V\left(\boldsymbol{w}^0, y^0\right)\right) > y\right) = \frac{\sum_{i \in B \setminus C\left(y^0, y, \boldsymbol{w}^0, \boldsymbol{w}\right)} exp\left(\mathrm{v}_i\left(w_i^0, y^0\right)\right)}{\sum_{k \in B} exp\left(max\left(\mathrm{v}_k\left(w_k, y\right), \mathrm{v}_k\left(w_k^0, y^0\right)\right)\right)}$$

*for* $y \ge 0$.

From eq. (8) and Corollary 8 we obtain that

$$\text{(11)} \quad \mathrm{E}\,\mathrm{Y_B}\left(\mathbf{w}, \mathrm{V}\left(\mathbf{w}^0, \mathrm{y}^0\right)\right) = \sum_{\mathrm{i} \in \mathrm{B}} \exp\left(\mathrm{v_i}\left(\mathrm{w_i^0}, \mathrm{y}^0\right)\right) \int_0^{\mathrm{y_{ii}}\left(\mathrm{w_i^0}, \mathrm{y}^0, \mathrm{w_i}\right)} \frac{\mathrm{dy}}{\sum_{\mathrm{k} \in \mathrm{B}} \exp\left(\max\left(\mathrm{v_k}\left(\mathrm{w_k}, \mathrm{y}\right), \mathrm{v_k}\left(\mathrm{w_k^0}, \mathrm{y}^0\right)\right)\right)}.$$

**Corollary 9**

*Suppose* $\varepsilon_0, \varepsilon_1, ..., \varepsilon_M$ *, are independent and extreme value distributed. Then*

$$P\left(J_B^*\left(\boldsymbol{w}, \boldsymbol{w}^0, y^0\right) = j \middle| Y_B\left(\boldsymbol{w}, V\left(\boldsymbol{w}^0, y^0\right)\right) = y\right) = P_j\left(C\left(y^0, y, \boldsymbol{w}^0, \boldsymbol{w}\right), \boldsymbol{w}, y\right) = \frac{exp\left(\mathrm{v}_j\left(w_j, y\right)\right)}{\sum_{k \in C\left(y^0, y, \boldsymbol{w}^0, \boldsymbol{w}\right)} exp\left(\mathrm{v}_k\left(w_k, y\right)\right)}$$

*for* $j \in B$ *and* $y \ge y_{jj}\left(w_j^0, y^0, w_j\right)$.

A proof of Corollary 9 is given in the appendix.

We realize that the results of Corollaries 5 to 9 are quite tractable from a computation viewpoint.

From Corollary 9 the next result is immediate.

**Corollary 10**

*Assume that* $\varepsilon_0, \varepsilon_1, ..., \varepsilon_M$ *, are independent and extreme value distributed. Then for* $y \geq y_{jj}\left(w_j^0, y^0, w_j\right)$

$$P\left(J_B^*\left(\boldsymbol{w}^0, y^0, \boldsymbol{w}\right) = j\right) = \int_0^\infty P_j\left(C\left(y^0, y, \boldsymbol{w}^0, \boldsymbol{w}\right), y, \boldsymbol{w}\right) P\left(Y_B\left(\boldsymbol{w}, V\left(\boldsymbol{w}^0, y^0\right)\right) \in dy\right)$$

*and the distribution of* $Y_B\left(\boldsymbol{w}, V\left(\boldsymbol{w}^0, y^0\right)\right)$ *is given by Corollary 8.*

**Example**

Consider a nested logit model with 4 alternatives where

$$G\left(x_0, x_1, x_2, x_3\right) = e^{-x_0} + e^{-x_1} + \left(e^{-x_2/\theta} + e^{-x_3/\theta}\right)$$

where $\theta \in (0,1]$ and $1-\theta^2$ has the interpretation as the correlation between the error terms of the utilities of alternatives two and three. We then get

$$\frac{G_i\left(-h_0, -h_1, -h_2, -h_3\right)}{G\left(-h_0, -h_1, -h_2, -h_3\right)} = \frac{e^{h_i}}{e^{h_0} + e^{h_1} + \left(e^{h_2/\theta} + e^{h_3/\theta}\right)}$$

for $i \in \{0,1\}$, and

$$\frac{G_i\left(-h_0, -h_1, -h_2, -h_3\right)}{G\left(-h_0, -h_1, -h_2, -h_3\right)} = \frac{\left(e^{h_2/\theta} + e^{h_3/\theta}\right)^{\theta-1} e^{h_i/\theta}}{e^{h_0} + e^{h_1} + \left(e^{h_2/\theta} + e^{h_3/\theta}\right)^{\theta}}$$

for $i \in \{2,3\}$, where $h_j$ is given in Theorem 4. By Corollaries 5 and 6 we can calculate the corresponding distributions of the expenditure function and initial choice.

# 5. Conclusion

In this paper we have demonstrated that the notion of random expenditure function and compensated choice probabilities can be readily adapted within a discrete choice setting. When the model belongs to the Generalized Extreme Value (GEV) class then the formulae for the compensated choice probabilities simplifies. Since Dagsvik (1995) has demonstrated that a general random utillity model can be approximated arbitrarily closely by models belonging to the GEV family, the GEV setting represents no loss of generality. We have also demonstrated that one can obtain analytic formulae for the distribution of Compensating Variation. Also in this case the formulae simplifies when the model belongs to the GEV class. Although we have focused on Compensating Variation in this paper, the derivation of the distribution of Equivalent Variation is completely analogous.

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## Appendix

**Proof of Theorem 1:**

Since

$$\max_{k\in B}\left(v_k\left(w_k, y\right)+\varepsilon_k\right)$$

is increasing in y with probability one, we get that

$$P\left(Y_B(\mathbf{w},u)>y\right)=P\left(\max_{k\in B}\left(v_k\left(w_k,y\right)+\varepsilon_k\right)<u\right)$$

$$=P\left(\bigcap_{k\in B}\left(v_k\left(w_k,y\right)+\varepsilon_k<u\right)\right)$$

$$=F^B\left(u-v_0\left(w_0,y\right),u-v_1\left(w_1,y\right),\ldots,u-v_m\left(w_m,y\right)\right).$$

Q.E.D.

**Proof of Theorem 2:**

Due to the fact that

$$\left\{Y_B\left(\mathbf{w},y\right)=y\right\}\Leftrightarrow\left\{\max_{k\in B}\left(v_k\left(w_k,y\right)+\varepsilon_k\right)=u\right\},$$

we have that

$$\psi_j(y)\equiv P\left(v_j\left(w_j,y\right)+\varepsilon_j=\max_{k\in B}\left(v_k\left(w_k,y\right)+\varepsilon_k\right)\middle|Y_B\left(\mathbf{w},u\right)=y\right)$$

$$=P\left(v_j\left(w_j,y\right)+\varepsilon_j=u\middle|\max_{k\in B}\left(v_k\left(w_k,y\right)+\varepsilon_k\right)=u\right)$$

$$=\frac{F_j^B\left(u-v_0\left(w_0,y\right),u-v_1\left(w_1,y\right),\ldots,u-v_m\left(w_m,y\right)\right)}{\sum_{k\in B}F_k^B\left(u-v_0\left(w_0,y\right),u-v_1\left(w_1,y\right),\ldots,u-v_m\left(w_m,y\right)\right)}.$$

Hence

$$P_j\left(B,u,\mathbf{w}\right)=\int_0^\infty\psi_j(y)P\left(Y_B\left(\mathbf{w},u\right)\in dy\right)\ .$$

Hence, by inserting for $\psi_j(y)$ and the distribution of $Y_B(\mathbf{w},u)$ given in Theorem 1 we obtain the result of Theorem 2.

Q.E.D.

**Proof of Theorem 3:**

We have that

$$P_j^h(B,u,\mathbf{w}) = \int P\Big(v_j(w_j,y)+\varepsilon_j = \max_{k\in B}\big(v_k(w_k,y)+\varepsilon_k\big)\Big| Y_B(\mathbf{w},u)=y\Big)P\big(Y_B(\mathbf{w},u)\in dy\big).$$

Now recall that by Strauss (1979) and Lindberg et al. (1995) the multivariate extreme value distribution has the property that $\max_{k\in B}\big(v_k(w_k,y)+\varepsilon_k\big)$ is independent of which alternative that maximizes utility. Hence due to the fact that

$$\big\{Y_B(\mathbf{w},u)=y\big\} \Leftrightarrow \Big\{\max_{k\in B}\big(v_k(w_k,y)+\varepsilon_k\big)=u\Big\}.$$

we obtain that

$$\begin{aligned}&P\Big(v_j(w_j,y)+\varepsilon_j = \max_{k\in B}\big(v_k(w_k,y)+\varepsilon_k\big)\Big| Y_B(\mathbf{w},u)=y\Big)\\ &= P\Big(v_j(w_j,y)+\varepsilon_j = \max_{k\in B}\big(v_k(w_k,y)+\varepsilon_k\big)\Big)\end{aligned}$$

Thus, the result of Theorem 3 follows.

Q.E.D.

**Proof of Theorem 4:**

Let

(A.1) $$Z_1(y) = \max_{k\in C\left(y^0,y,\mathbf{w}^0,\mathbf{w}\right)}\big(v_k(w_k,y)+\varepsilon_k\big),$$

(A.2) $$Z_2(y) = \max_{k\in B\setminus C\left(y^0,y,\mathbf{w}^0,\mathbf{w}^0\right)}\big(v_k(w_k,y)+\varepsilon_k\big),$$

(A.3) $$Z_1^0 = \max_{k\in C\left(y^0,y,\mathbf{w}^0,\mathbf{w}^0\right)}\big(v_k(w_k^0,y^0)+\varepsilon_k\big)$$

and

(A.4) $$Z_2^0 = \max_{k \in B \setminus C\left(y^0, y, \mathbf{w}^0, \mathbf{w}^0\right)} \left( v_k \left( w_k^0, y^0 \right) + \varepsilon_k \right).$$

Note that $Z_1(y) \geq Z_1^0$ and $Z_2(y) < Z_2^0$ with probability one. Hence

(A.5) $$\left\{ Y_B \left( \mathbf{w}, V\left( \mathbf{w}^0, y^0 \right) \right) \leq y \right\} = \left\{ \max\left( Z_1(y), Z_2(y) \right) \geq \max\left( Z_1^0, Z_2^0 \right) \right\}$$
$$\Leftrightarrow \left\{ \left( Z_1(y) \geq \max\left( Z_1^0, Z_2^0 \right) \right) \cup \left( Z_2(y) \geq \max\left( Z_1^0, Z_2^0 \right) \right) \right\} \Leftrightarrow \left\{ Z_1(y) \geq Z_2^0 \right\}.$$

Similarly, we get that

(A.6) $$\left\{ Y_B \left( \mathbf{w}, V\left( \mathbf{w}^0, y^0 \right) \right) > y \right\} \Leftrightarrow \left\{ Z_1(y) < Z_2^0 \right\}.$$

Accordingly,

(A.7) $$\left\{ Y_B \left( \mathbf{w}, V\left( \mathbf{w}^0, y^0 \right) \right) = y \right\} \Leftrightarrow \left\{ Z_1(y) = Z_2^0 \right\}.$$

Furthermore,

$$\left\{ \left( J_B\left( \mathbf{w}^0, y^0 \right) = i \right) \cap \left( J_B^*\left( \mathbf{w}, \mathbf{w}^0, y^0 \right) = j \right) \cap \left( Y_B\left( \mathbf{w}, V\left( \mathbf{w}^0, r \right) \right) = y \right) \right\}$$
$$= \left\{ \left( J_B\left( \mathbf{w}^0, y^0 \right) = i \right) \cap \left( J_B\left( \mathbf{w}, y \right) = j \right) \cap \left( Z_1(y) = Z_2^0 \right) \right\}$$
$$= \left\{ \left( v_i\left( w_i^0, y^0 \right) + \varepsilon_i = \max\left( Z_1^0, Z_2^0 \right) \right) \cap \left( v_j\left( w_j, y \right) + \varepsilon_j = \max\left( Z_1(y), Z_2(y) \right) \right) \cap \left( Z_1(y) = Z_2^0 \right) \right\}.$$

Next note that when $Z_1(y) = Z_2^0$,

$$\max\left( Z_1^0, Z_2^0 \right) = \max\left( Z_1^0, Z_1(y) \right) = Z_1(y) = Z_2^0$$

and

$$\max\left( Z_1(y), Z_2(y) \right) = \max\left( Z_2^0, Z_2(y) \right) = Z_2^0 = Z_1(y).$$

Hence

(A.8) $$\left\{ \left( v_i\left( w_i^0, y^0 \right) + \varepsilon_i = \max\left( Z_1^0, Z_2^0 \right) \right) \cap \left( v_j\left( w_j, y \right) + \varepsilon_j = \max\left( Z_1(y), Z_2(y) \right) \right) \cap \left( Z_2^0 = Z_1(y) \right) \right\}$$
$$= \left\{ \left( v_i\left( w_i^0, y^0 \right) + \varepsilon_i = Z_2^0 \right) \cap \left( v_j\left( w_j, y \right) + \varepsilon_j = Z_1(y) \right) \cap \left( Z_2^0 = Z_1(y) \right) \right\}.$$

But (A.8) implies that for $i \neq j$

$$P\left(Y_B\left(\mathbf{w},V\left(\mathbf{w}^0,y^0\right)\right)\in\left(y,y+\Delta y\right),\ \ J_B\left(\mathbf{w}^0,y^0\right)=i,\ \ J_B^*\left(\mathbf{w},\mathbf{w}^0,y^0\right)=j\right)$$

$$=P\left(v_i\left(w_i^0,y^0\right)+\varepsilon_i=Z_2^0,\ \ v_j\left(w_j,y+\Delta y\right)+\varepsilon_j\ge Z_2^0\ge v_j\left(w_j,y\right)+\varepsilon_j,\max_{k\in C\left(y^0,y,\mathbf{w}^0,\mathbf{w}\right)\setminus\{j\}}\left(v_k\left(w_k,y\right)+\varepsilon_k\right)\le Z_2^0\right)+o\left(\Delta y\right)$$

$$=\int P\left(v_i\left(w_i^0,y^0\right)+\varepsilon_i\in du,\max_{k\in B\setminus\left(C\left(y^0,y,\mathbf{w}^0,\mathbf{w}\right)\setminus\{j\}\right)}\left(v_k\left(w_k^0,y^0\right)+\varepsilon_k\right)\le u,v_j\left(w_j,y+\Delta y\right)+\varepsilon_j\ge u\ge v_j\left(w_j,y\right)+\varepsilon_j,\right.$$

$$\left.\max_{k\in C\left(y^0,y,\mathbf{w}^0,\mathbf{w}\right)\setminus\{j\}}\left(v_k\left(w_k,y\right)+\varepsilon_k\right)\le u\right)+o\left(\Delta y\right)$$

$$=\int F_i^B\left(u-h_0\left(w_0^0,y^0,w_0,y\right),u-h_1\left(w_1^0,y^0,w_1,y\right),...,u-h_m\left(w_m^0,y^0,w_m,y\right)\right)du$$

$$-\int F_i^B\left(u-h_0\left(w_0^0,y^0,w_0,y+\Delta y\right),u-h_1\left(w_1^0,y^0,w_1,y+\Delta y\right),...,u-h_m\left(w_m^0,y^0,w_m,y+\Delta y\right)\right)du$$

$$=o\left(\Delta y\right)+\left(v_j\left(w_j,y+\Delta y\right)-v_j\left(w_j,y\right)\right)\int F_{ij}^B\left(u-h_0\left(w_0^0,y^0,w_0,y\right),u-h_1\left(w_1^0,y^0,w_1,y\right),...,u-h_m\left(w_m^0,y^0,w_m,y\right)\right)du$$

which yields

(A.9)
$$P\left(Y_B\left(\mathbf{w},V\left(\mathbf{w}^0,y^0\right)\right)\in dy,J_B\left(\mathbf{w}^0,y^0\right)=i,J_B^*\left(\mathbf{w},\mathbf{w}^0,y^0\right)=j\right)$$
$$=v_j\left(w_j,dy\right)\int F_{ij}^B\left(u-h_0\left(w_0^0,y^0,w_0,y\right),u-h_1\left(w_1^0,y^0,w_1,y\right),...,u-h_m\left(w_m^0,y^0,w_m,y\right)\right)du$$

for $i\in B\setminus C\left(y^0,y,\mathbf{w}^0,\mathbf{w}\right)$ and $j\in C\left(y^0,y,\mathbf{w}^0,\mathbf{w}\right)$, and zero for $i\notin B\setminus C\left(y^0,y,\mathbf{w}^0,\mathbf{w}\right)$, $j\notin C\left(y^0,y,\mathbf{w}^0,\mathbf{w}\right)$, $i\ne j$.

For $j=i$ it follows that

(A.10)
$$P\left(Y_B\left(\mathbf{w},V\left(\mathbf{w}^0,y^0\right)\right)=y,J_B\left(\mathbf{w}^0,y^0\right)=J_B^*\left(\mathbf{w},\mathbf{w}^0,y^0\right)=i\right)$$
$$=P\left(v_i\left(w_i^0,y^0\right)=v_i\left(w_i,y_{ii}\left(w_i^0,y^0,w_i\right)\right),v_i\left(w_i^0,y^0\right)+\varepsilon_i>\max_{k\in B\setminus\{i\}}\left(v_k\left(w_k^0,y^0\right)+\varepsilon_k\right)\right)$$
$$=P\left(J_B\left(\mathbf{w}^0,y^0\right)=i\right)$$

for $y=y_{ii}\left(w_i^0,y^0,w_i\right)$. Moreover, the expenditure function in this case has measure zero outside the point $y_{ii}\left(w_i^0,y^0,w_i\right)$.

Q.E.D.

**Proof of Corollary 7:**

Let $Z_1(y)$, $Z_2(y)$ and $Z_2^0$ be defined as in (A.1), (A.2) and (A.4), respectively. As in the proof of Theorem 4 it follows that

$$P\left(J_B^*\left(\mathbf{w}, \mathbf{w}^0, y^0\right) = j \mid Y_B\left(\mathbf{w}, V\left(\mathbf{w}^0, y^0\right)\right) = y\right)$$
$$= P\left(v_j\left(w_j, y\right) + \varepsilon_j = Z_1(y) \mid Z_1(y) = Z_2^0\right).$$

Since the random terms are independent it follows that $\varepsilon_j$ is independent of $Z_2^0$ and $Z_1(y)$ and $Z_2^0$ are independent.

Moreover, due to properties of the extreme value distribution it follows that for given y

$$P\left(v_j\left(w_j, y\right) + \varepsilon_j = Z_j(y) \mid Z_j(y)\right) = P\left(v_j\left(w_j, y\right) + \varepsilon_j = Z_j(y)\right).$$

But then it follows that

$$P\left(v_j\left(w_j, y\right) + \varepsilon_j = Z_j(y) \middle| Z_j(y) = Z_2^0\right) = P\left(v_j\left(w_j, y\right) + \varepsilon_j = Z_j(y)\right) = P\left(J_{C\left(y^0, y, \mathbf{w}^0, \mathbf{w}\right)}\left(\mathbf{w}, y\right) = j\right).$$

Q.E.D.