Discussion Papers No. 247, January 1999
Statistics Norway, Research Department

Rune Johansen and John K. Dagsvik

The Dynamics of a Behavioral Two-Sex Demographic Model

Abstract:
In this paper, we examine the dynamic properties of a particular demographic model. An essential part of the model is the marriage function which is derived from assumptions about the behavior of women and men in a market where each individual is looking for a suitable partner. By means of simulation experiments we investigate different aspects of the model. Specifically, we find that it is difficult to determine parameters related to preferences, birth and death rates, such that a non-trivial stable equilibrium is attained.

Keywords: Two-sex demographic models, Marriage function, Birth rates, Non-trivial equilibria

JEL classification: C78, J11

Address: John K. Dagsvik, Statistics Norway, Research Department. E-mail: john.dagsvik@ssb.no

Discussion Papers comprise research papers intended for international journals or books. As a preprint a Discussion Paper can be longer and more elaborate than a standard journal article by including intermediate calculation and background material etc.

Abstracts with downloadable PDF files of
Discussion Papers are available on the Internet: http://www.ssb.no

For printed Discussion Papers contact:

Statistics Norway
Sales- and subscription service
P.O. Box 1260
N-2201 Kongsvinger

Telephone: +47 62 88 55 00
Telefax: +47 62 88 55 95
E-mail: Salg-abonnement@ssb.no

1 Introduction

The use of mathematical models in human demography dates back to the 1920's. In classical studies the models were linear (cf. Leslie [1945]). However, these classical models ignore important aspects of the reproduction process, the main problem being that they are based on age specific fertility and death rates for *females only*. In other words, the *mating process*, that is, the forming of marriages, is ignored. This process plays a crucial role in the reproduction of the human population, as the number of births is dependent on the number of marriages. The number of marriages in each combination of age-groups of males and females is affected by the total number of individuals in these age-groups, and hence the number of births may depend on the size and age-structure of both the (mature) male and female population in a non-trivial way. Thus, a realistic population theory should incorporate a model that predicts how marriages are formed.

The recognition of this fact has lead to several attempts to formulate two-sex models, see for example Pollard [1995] and the references therein. Unfortunately, the two-sex models proposed in the literature suffer from a fundamental weakness in that the associated marriage models are not derived explicitely from behavioral principles, although they are constructed so as to fulfill particular reasonable qualitative properties based on biological and demographic considerations. Thus, from a theoretical viewpoint these models are somewhat *ad hoc*.

This paper differs from previous analyses of two-sex models in that our point of departure is a particular behavioral marriage model proposed by Dagsvik [1998] and Dagsvik et.al. [1998]. Given this marriage model, the updating is described by the standard Markovian schedule. But, in contrast to the classical models, it now follows that the model becomes nonlinear. In general, such models can be very hard to analyze. However, during the last few decades the mathematical theory of nonlinear dynamical systems has provided us with a powerful apparatus that may be useful for revealing some of the structural properties of such models.

The paper is organized as follows: First we give a short survey on demographic models, and the qualitative properties of such models. Thereafter we give a brief presentation and discussion of the marriage model due to Dagsvik [1998]. Based on this model, we derive a demographic model for the number of women and men in specific population groups at a given time, and examine the dynamical properties of this particular model.

2 Demographic Population models

We will at first give a short summary on demographic models and the two-sex problem. For a more extensive review we refer to Pollard [1995]. The first models considered in the literature were *one-sex* models, based on female reproduction rates. Such models may work well if the population of men and women in each age group are of similar sizes, but may yield rather poor results in the case of imbalances between the population sizes of men and women, cf. Pollak [1990].

2.1 The Leslie model

In classical stable population theory the female population (at time t) is represented by a vector

$$F(t) = (F_1(t), ..., F_n(t))$$

where $F_i(t)$ is the number of females of age i at time t. The description of how the population evolves over time has two 'building blocks'; namely a vector of *survival rates* $\sigma = (\sigma_1, ..., \sigma_n)$, where $\sigma_n = 0$ (n is the maximum age of an individual in the population), and birth or *fertility* rates described by a vector $\phi = (\phi_1, ..., \phi_n)$. Thus the number of newborn (females) at time $t+1$ may be represented by a linear combination of the $F_i(t)$'s the following way:

$$F_1(t+1) = \sum_{i=1}^{n} \phi_i F_i(t).$$

Furthermore, the population is updated according to the (ageing) relation

$$F_i(t+1) = \sigma_{i-1} F_{i-1}(t)$$

for $2 \leq i \leq n$. Hence, in matrix notation we have

$$F(t+1) = LF(t) \tag{2.1}$$

where

$$L = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{n-1} & \phi_n \\ \sigma_1 & 0 & \cdots & 0 & 0 \\ . & . & & . & . \\ . & . & & . & . \\ 0 & 0 & \cdots & \sigma_{n-1} & 0 \end{bmatrix}$$

The matrix L is called the *Leslie-matrix* named after P.H.Leslie who was a pioneer on this subject, cf. Leslie [1945]. In the original version of the Leslie model, L is assumed to be constant. If the population converges towards a constant level, we say that the matrix equation (2.1) has a (stable) *equilibrium*. If the population return to an earlier state after a finite number of generations we say that the equation has a *periodic orbit*. In some circumstances (particularily those cases where no equilibria or periodic orbits are present) it may also be of great interest to detect whether the population grows at a constant rate or not. This issue has been investigated by Keyfitz [1972] and others.

Mathematically, there is no big difference between fixed and periodic points. If the population returns to an earlier state after a finite number of generations it has a periodic orbit. A fixed point is a periodic point of period 1. In the case of the Leslie model, periodic orbits satsify the equation

$$L^p F = F$$

where $p \geq 1$ denotes the period. Since the matrix L is non-negative, the properties of the periodic points are described by a theorem due to Perron & Frobenius (cf. Keyfitz [1972]). Specifically, there are two possibilities: If all the eigenvalues of L are real, the long term behavior of the system is described by the eigenvalue λ_0 having the largest absolute value. If $\lambda_0 < 1$, the population will eventually become extinct. If $\lambda_0 > 1$, the population will grow towards infinity with a constant rate equal to λ_0. In the case $\lambda_0 = 1$, the population will converge towards a stable equilibrium. In the case of complex eigenvalues (which all have to occur in conjugated pairs since L is real), the limit behavior of the system is a periodic orbit, with period equal to the number of complex eigenvalues plus one, that is, the period has to be an odd number.

Notice that the trivial (and stable) equilibrium $F = 0$ is always a solution of the equation $LF = F$. In one-sex models this trivial equilibrium $F = 0$ is 'uninteresting', while in the two-sex case to be considered next, the existence of trivial solutions usually makes the task of finding non-trivial equilibria by means of fixed point theorems more difficult.

2.2 Two-sex models

In one-sex models the number of offspring is only dependent on the number of females. In many cases this may seem like a plausible assumption, as, at least in theory, even one individual male can produce enough sperm to impregnate millions of females. This particular setting is consistent with *female dominance*, namely that there are always enough males to fertilize all females. For species where only a handful of males is sufficient to ensure a successful reproduction, the fate of most of the males is in this context unimportant. However, in populations with monogamy, as in most human societies, the role of the mating process becomes important.

In populations without female dominance a common phenomenon is *marriage squeeze*, in which the reproduction is limited by the availability of the scarcer sex. This phenomenon is well known even in human populations despite the fact that the overall sex ratio never deviates far from unity. Patterns of preferences for age, education, etc., may also contribute to the marriage squeeze. Thus, in populations without female dominance, both sexes must be incorporated in order to provide an appropriate representation of the population dynamics.

The main difference between one-sex and two-sex models (except from the introduction of the second sex) is that the birth and survival schedules no longer are assumed to be constant, but depend on the size of the population, and its age-sex composition. The number of births is dependent on the number of marriages, and the number of offspring produced by a married female may not only depend on her age, but also on the age of her mate. In addition, the behavior of males and females in the marriage market is dependent on the size of the respective age classes of (single) men and women.

This implies a *non-linear* model, in which the *mating rule*, describing how marriages are formed, becomes an important element. Caswell and Weeks [1986] studied a two sex model under particular assumptions about the mating behavior. In fact, they analyzed several possible forms of the mating function. Chung [1994] extended the study by Caswell and Weeks, and made a more thorough analysis of the dynamics, showing that interesting dynamical behavior may occur also at "realistic" parameter levels, in contrast to Caswell and Weeks who used rather extreme parameter values.

We will now describe the two-sex modelling framework formally. As in one-sex models the population is divided into n age groups or *categories*. The population at time t may be described by a $(2n)$-vector $(M(t), F(t))$, where

$$M(t) = (M_1(t), ..., M_n(t))$$

represents the male population, and as above

$$F(t) = (F_1(t), ..., F_n(t))$$

represents the female population. The number of births is dependent on the number of marriages. In traditional demographic studies one has usually assumed that the mating function has a particular form, based on different biological considerations (see Caswell and Weeks [1986]), in contrast to the present approach, which is, as mentioned above, based on a particular behavioral marriage model to be described in section 3.

Let ϕ_{ij} be the expected number of *female offspring* of a male in age group i married to a female from age group j. If now ν is the rate of male to female newborns (the *sex ratio*, assumed to be constant), then $\nu\phi_{ij}$ is the expected number of *male* offspring produced by a couple where the male has age i and the female has age j. Now let $\mu(M,F)$ denote the *mating rule* or *marriage function*, that is, the matrix function predicting the number of marriages in each age combination as a function of the number of single men and women in each age group, represented by the vectors M and F. Entry (i,j) in the matrix $\mu(M,F)$ is denoted by $\mu_{ij}(M,F)$, and is equal to the number of marriages between males in age group i and females in age group j. As indicated above, the functions $\{\mu_{ij}\}$ may be non-linear. From the above considerations the number of newborn at time $t+1$ may be expressed as:

(2.2) $$F_1(t+1) = \sum_{ij} \phi_{ij}\mu_{ij}(M(t),F(t))$$

and

$$M_1(t+1) = \nu F_1(t+1).$$

The agening of the population follows from the (linear) recursion formula:

(2.3) $$M_i(t+1) = \sigma^M_{i-1} M_{i-1}(t)$$

for $i = 2, ..., n$, and

(2.4) $$F_j(t+1) = \sigma^F_{j-1} F_{j-1}(t),$$

for $j = 2, ..., n$, where σ^M_i, σ^F_j are the survival rates of males of age i and females of age j, respectively. The above relations define a vector function $\boldsymbol{g} = (g_1, ..., g_n, g_{n+1}, ..., g_{2n})$ by

$$M_i(t+1) = g_i(M(t),F(t))$$

for $i = 1, ..., n$, and

$$F_j(t+1) = g_{n+j}(M(t),F(t))$$

for $j = 1, ..., n$. Hence we get a dynamical system where the population is updated according to the recursive equation

$$(M(t+1), F(t+1)) \quad = \quad \boldsymbol{g}(M(t), F(t)). \tag{2.5}$$

Due to certain biological and mathematical considerations, several authors have suggested that the mating function $\mu(.)$ should satisfy a number of criteria or axioms, including the following (see McFarland [1972] and Pollard [1995]):

A1. $\mu(M, F)$ is defined for all (non-negative) vector combinations (M, F).

A2. $\mu(M, F) \geq 0$ for all $M \geq 0, \ F \geq 0$.

A3. $\sum_i \mu_{ij}(M, F) \leq F_j$ and $\sum_j \mu_{ij}(M, F) \leq M_i$. The number of marriages involving members of one category can not exceed the total number of members in that category.

A4. The number of marriages should depend heavily on the ages of the males and females.

A5. μ_{ij} is non-decreasing in M_i and F_j, and strictly increasing for some values of M_i and F_j (A larger population yields more marriages than a smaller one).

A6. μ_{ij} is *non-increasing* (and over some interval a strictly decreasing) function of $M_r, F_s, r \neq i, s \neq j$.

A7. The negative effect on μ_{ij} of an increase in M_s should be greater than the negative effect on μ_{ij} of an equivalent increase in M_r if s is closer to i than r is. Likewise with the sexes interchanged.

A8. $\mu(M, 0) = \mu(0, F) = 0$. The extinction of one sex inevitably rules out the possibility of a marriage, eventually making the population extinct.

A9. μ is continuous in M and F (some authors assume the mating function to be defined only on the integers. However, mathematically, it may be convenient to extend the definition of the mating function to the positive real numbers as well).

A10. $\mu(\lambda M, \lambda F) = \lambda \mu(M, F)$ (homogenity).

In most papers on two-sex models, the mating function is assumed to be on a particular closed form (see for example Caswell and Weeks [1986] or Pollard [1995]). Typical explicit function forms that have beeen applied in the particular case with no age structure are summarized in the following table:

Table1: Different marriage models discussed in the literature

$\mu(M,F)$	*interpretation*
F	Female dominance.
M	Male dominance.
$aM + (1-a)F$, where $0 < a < 1$.	Weighted mean.
$k\frac{M+F}{2}$.	Arithmetic mean.
$k(M \cdot F)^{\frac{1}{2}}$.	Geometric mean.
$\frac{2kMF}{M+F}$.	Harmonic mean.
$k \min\{M,F\}$.	Minimum.

In this table $k > 0$ is a suitable real constant to be determined (by data). Notice that in the case where $\mu(M,F) = k\min\{M,F\}$ (minimum), there is a one-to-one correspondence between the number of marriages (births) and the availability of the scarcer sex. As mentioned by Pollard [1995], most of these functions have serious flaws, and Pollard finds the harmonic mean to be the most interesting. The two-sex model examined by Caswell and Weeks [1986] and Chung [1994] was based on this mating function. In the general case with age-structured populations, Pollard [1995] and others have proposed the following extension of the harmonic mean function, namely

$$\mu_{ij}(M,F) = \frac{\kappa_{ij} \cdot M_i \cdot F_j}{\sum_r \gamma_{ri} M_r + \sum_r \theta_{rj} F_r},$$

where $\{\kappa_{ij}\}, \{\gamma_{ri}\}$ and $\{\theta_{rj}\}$ are parameters. The main weakness of all these functions is that they are not derived from a theory about individual behavior. In other words, they are ad hoc from a theoretical point of view. As mentioned above, our aim in this paper is to investigate the dynamical properties of the above two-sex model when the mating rule (marriage model) is based on a particular behavioral theory, to be introduced below. However, before we present our marriage model, we shall give a brief survey of some relevant material from the theory of dynamical systems.

2.3 A short review of some aspects of the theory of dynamical systems that are relevant when studying two-sex models

An important purpose when analyzing dynamical systems is to reveal the long term or *asymptotic* behavior of the system. In particular, it is of interest to examine the structure of the *fixed* and *periodic points* (*equilibria*), a task which may be difficult. There are two types of equilibria that are found interesting in (human) demography. One is the case of a constant growth of the population, while the other is the case of the population remaining unchanged over time. The last case is called a *proper* equilibrium. If the population returns to an earlier state after a finite number of generations we say that it possesses a periodic orbit. Both proper equilibria and periodic orbits (and other interesting dynamical phenomena as well) have been observed in many animal populations, while among human beings, it seems like most populations grow constantly. But even if we are not able to control the population size, it may be of great importance to understand to which extent the structural parameteres affect the growth of the population. Hence, in many circumstances, the conditions for a 'constant growth' equilibrium may be the most interesting.

The trivial equilibrium is always a possibility in (realistic) demographic models (if the population enters the state of extinction, then it will remain extinct forever). This may complicate the analysis, since the model can still have non-trivial equilibria which may be hard to find, especially when these equilibria are *unstable*.

In the nonlinear case, one may sometimes generalize the techniques provided by the Perron-Frobenius theorem. The main idea is first to detect (all the) fixed and periodic points of the map describing the system. Thereafter, the *linearization* of the map, that is, the Jacobian matrix of the map evaluated at the fixed or periodic point, is computed. The dynamics of the model in a neighborhood of the equilibrium is determined by the *spectrum* (the set of eigenvalues) of the linearization. This is due to the following theorem (cf. Hartman [1964]):

Theorem (Hartman-Grobman): *In a neighborhood of a hyperbolic fixed (periodic) point a dynamical system is topologically conjugated to its linearization, determined by its Jacobian matrix evaluated at the fixed or periodic point.*

Remark: Two dynamical systems are called *topologically conjugated* if their fixed points (equilibria) and periodic orbits have the same structure. A fixed (periodic) point of a linear system is called *hyperbolic* whenever none of its eigenvalues have absolute value (modulus) equal to one, that is, none of the eigenvalues are lying on the unit circle in the complex plane. Thus, in a neighborhood of a hyperbolic fixed point p, every non-linear map $\boldsymbol{g}$ may be approximated by the linear map $x \mapsto D\boldsymbol{g}(p)x$, where $D\boldsymbol{g}(p)$ denotes the Jacobian matrix of $\boldsymbol{g}$ evaluated at p. The dynamics of this linear system may be analyzed by traditional eigenvalue analysis. In practice one may not know whether the fixed point is hyperbolic or not, but by computing the Jacobian, and finding its eigenvalues, one may conclude that hyperbolicity of the linearization must imply hyperbolicity of the original system, and vice versa.

In the nonlinear case, the number of possible combinations of the eigenvalues is in general large, depending on the dimension of the model (number of age groups). The *invariant manifold theorem* (cf. Hirsch et.al. [1977]) tells us that the map defining the model is contracting or expanding in the direction of an eigenvector according to whether the corresponding eigenvalue has absolute value smaller or greater than one. The behavior of the model will also vary, depending on whether this eigenvalue is real or complex. Complex eigenvalues always occur in conjugated pairs since the Jacobian is a real matrix. The case of a real eigenvalue of multiplicity larger than one must also be explicitely treated. If the absolute value of an eigenvalue is equal to one, the map is neither contracting nor expanding along the corresponding eigenvector, and we say that the map possesses a *center manifold* (see for example Guckenheimer and Holmes [1983]).

The (general) two-sex modeling framework outlined above, is dependent on a parameter set including the birth rates ϕ_{ij}, the survival rates σ_i, and the sex ratio at birth, ν. When the parameters vary in a domain, the dynamical behavior of the model may change. Parameter values at which such a change take place are called *bifurcation points*, and the process the system undergoes at such a point is called a *bifurcation*. In theory, there are several types of bifurcations a dynamical system may undergo as the parameters vary. It is an interesting, but in general very difficult task to classify these.

During the last couple of decades much attention has been given to the possibility of a dynamical system becoming *chaotic*. Loosely speaking, this means that all of the equilibria becomes unstable, and that the system becomes sensitive to initial conditions, making it impossible to predict future population sizes. Numerical simulations using the Caswell-Weeks model show that large enough values of the parameters can destabilize the equilibrium in age and sex structure, making their model chaotic (cf. Chung [1994]).

3 A two-sex marriage model derived from a particular matching game

In this section we shall discuss a particular model derived from assumptions about the behavior in the *marriage market*. In this market each man and woman (*agent*) is assumed to behave according to specific rules as follows. Each man and woman are supposed to have sufficient information about the potential partners so as to be able to establish a preference list, that is, a list which ranks all potential partners, including the alternative of being single. The matching process towards equilibrium takes place in several stages. There are no search costs, and the men and women have no information about the preferences of potential partners, which means that they are ignorant about their own chances in the market. Either the women or the men make offers, that is, if the men make the offers, no woman is allowed to make an offer and vice versa. A man is *acceptable* to a woman if the woman prefers to be matched to that particular man rather than staying single. A matching between a male and female who are not mutually acceptable, which means that at least one of the agents would prefer to be single rather than be matched to the other, is said to be *blocked by the unhappy agent.* A matching μ such that there exist a male and female who are matched to each other, but who prefer each other to their assignment at μ, given the rules of the game, is said to *block the matching* μ. We say that a matching μ is *stable* if it is not blocked by any individual or pair of agents.

Gale and Shapley [1962] (cf. Roth and Sotomayor [1990]) have demonstrated that stable matchings exist for every matching market. Specifically, they proved that the so-called *deferred acceptance procedure* produces a stable matching for any set of preferences provided the ordering of the preferences are *strict*, that is, indifferences are ruled out. This algorithm goes as follows: Suppose the men make the offers. First each man make an offer to his favorite woman. Thus a woman may receive offers from one or several men, or may receive none offers at all. Each woman immediately rejects the offer from any man who is unacceptable to her, and she rejects all but her most preferred among the acceptable offers too. Any man whose offer is not rejected at this point is kept temporarily 'engaged' until better offers arrive. At any step any man who was rejected at the previous step makes an offer to his next choice, that is, to his most preferred woman among those who have not yet rejected him. Each woman receiving offers rejects any from unacceptable men, and also rejects all but her most preferred among the new offers and any man she may have kept engaged from the previous step. The game is terminated after any step in which no man is rejected. The matches are now consummated with each man being matched to the woman he is engaged.

Based on the deferred acceptance algorithm Dagsvik [1998] obtained an aggregate model, that is, a model for the number of marriages between men and women in each age group. We shall now give a brief presentation of Dagsvik's model. For a more detailed presentation and proofs, we refer to Dagsvik [1998].

We assume the preferences of the males and females are represented by latent utility indicies. Now, let M_i, $i = 1,..,n$, be the number of (single) men in age group i, and $F_j, j = 1,...,n$, the number of (single) women in age group j. We define U_{ij}^{mf} to be the utility of male m in age group i of being married to female f in age group j. U_{i0}^{m} is the utility of male m in age group i of being single. Similarly, let U_{ji}^{fm} be the utility of female f of age group j of being matched to male m in age group i, and U_{j0}^{f} the utility of female f in age group j of being single. The utility functions are assumed to have the structure

$$U_{ij}^{mf} = a_{ij}\eta_{ij}^{mf},\ U_{i0}^{m} = a_{i0}\eta_{i0}^{m}$$

$$U_{ji}^{fm} = b_{ji}\eta_{ji}^{fm},\ U_{j0}^{f} = b_{j0}\eta_{j0}^{f}$$

where a_{ij}, b_{ji} are positive (non-negative) deterministic terms (*preference parameters*), and $\eta_{ij}^{mf}, \eta_{i0}^{m}, \eta_{ji}^{fm}, \eta_{j0}^{f}$, are positive random variables which are supposed to account for unobservables that affect the preferences. Without loss of generality we may 'normalize' the preference parameters for being single, that is, we let

$$a_{i0} = b_{j0} \equiv 1.$$

The random terms are assumed to be distributed according to the type I *extreme value distribution*, with cumulative distribution function given by

$$P(\eta_{ij}^{mf} \leq y) = P(\eta_{i0}^{m} \leq y) = P(\eta_{ji}^{fm} \leq y) = P(\eta_{j0}^{f} \leq y) = exp(-1/y)$$

for $y > 0$. The extreme value distribution is of particular interest in this context because it can be given a behavioral justification, and it is also tractable as it yields simple functional forms.

Given the above structure of the utility functions, Dagsvik [1998] demonstrates that the asymptotic number of marriages between males in age group i and females in age group j; X_{ij}, can be expressed as

$$X_{ij} = \frac{c_{ij} M_i F_j}{A_i B_j} \tag{3.1}$$

where A_i and B_j are determined by the follwong system of equations

$$A_i = 1 + \sum_k \frac{c_{ik} F_k}{B_k} \tag{3.2}$$

and

$$B_j = 1 + \sum_k \frac{c_{kj} M_k}{A_k} \tag{3.3}$$

for $i = 1, ..., n$, and $j = 1, ..., n$, and where $c_{ij} = a_{ij} b_{ji}$. The respective number of single males and females are given by

$$X_{i0} = \frac{M_i}{A_i} \tag{3.4}$$

and

$$X_{0j} = \frac{F_j}{B_j}. \tag{3.5}$$

From the above expressions (3.1),(3.2) and (3.3), we may derive a polynomial equation in X_{ij} of a degree dependent on the number of categories. Dagsvik [1998] demonstrated (by means of traditional fixed point techniques) that the system of equations (3.2) and (3.3) always has a unique real and positive solution. However, expressing this solution on a closed form is impossible in the general case. But, using numerical techniques it is straight forward to solve these equations.

Dagsvik et.al. [1998] investigated whether or not the above marriage model satisfies the Axioms A1-A7. Unfortunately, they were not able to prove whether or not A5 and A7 hold in the general case. They also found that, in general, A6 does not hold. However, for their particular estimates of the preference parameters, they did not find any case where A1-A7 were violated. From the expression (3.1) it is also evident that Axiom A8 is satisfied, and extending this formula to the real numbers makes the model continuous (differentiable) as well (Axiom A9). However, since A_i and B_j are dependent on the size of the male and female populations, the model is not homogenous, that is, Axiom A10 is violated.

In the special case where $n = 1$, that is, where there is only one category of males and females, (3.1) reduces to

$$X = \frac{1}{2}[\alpha\beta + M + F - \sqrt{(\alpha\beta + M + F)^2 - 4MF}\,], \tag{3.6}$$

where $\alpha = 1/a$, $\beta = 1/b$.

The above model (3.1) to (3.5) for the asymptotic number of marriages may be generalized by including the possibility of (feasible) *contracts.* A contract represents an agreement between the agents when forming a marriage. In the present context, important contract terms may for example be different residential locations. In the presence of flexible contracts, the quantity c_{ij} is modified to:

$$c_{ij} = \sum_{\omega} a_{ij}(\omega) b_{ji}(\omega) \tag{3.7}$$

with $a_{ij}(\omega)$ and $b_{ji}(\omega)$ being the preference parameters of the men and women respectively, under the contract ω. For a more precise description of this case, we refer to Dagsvik [1998].

4 Properties of the demographic model based on our marriage model

We will now examine the dynamics of a two-sex model of the form (2.5) based on the above marriage model, that is, the asymptotic number of matches X_{ij} represents the mating function μ_{ij} in the expression (2.2). Thus, if $(M(t), F(t))$ is the (mature) population at time t, the number of newborn females at time $t + 1$ can be expressed as

$$F_1(t+1) = \sum_{ij} \phi_{ij} X_{ij}(M(t), F(t)).$$

To study the properties of the above behavioral marriage model, it is desirable to find realistic values of the preference parameters a_{ij} and b_{ji} (and the birth and survival rates ϕ_{ij} and σ_i^F and σ_j^M as well). However, from the purpose of assessing the qualitative properties of the model, the choice of $\{a_{ij}\}$ and $\{b_{ji}\}$ may not be so critical. Our main purpose in this paper is not to utilize the model to provide practical predictions, but to achieve a better understanding of the dynamics of the model. When modeling human populations, the assumption of *one-year* age groups lead to huge models. In our analysis we have, for simplicity reasons, only considered the case where the number of age groups is equal to four and ten, respectively.

To detect (non-trivial) fixed and periodic points (of a map $\boldsymbol{g}$), we must solve an equation of the form

$$\boldsymbol{g}^n(x) = x \tag{4.1}$$

where $\boldsymbol{g}^n$ denotes the composition of $\boldsymbol{g}$ with itself n times. However, solving such equations are not always possible using analytical techniques, and numerical methods may fail as well if the fixed point is unstable (or semistable), that is, if the function $\boldsymbol{g}$ defining the system is not contracting along all the eigenvectors. The possibility of several equilibria makes the analysis even more complicated. Small variations in the model parameter may change the system from converging towards the trivial equilibrium, to a system where each orbit apparently tends towards infinity, possibly indicating that if non-trivial equilibria or periodic points exist, they are not stable. In such cases it may be more fruitful to go for an alternative strategy, e.g. to use the possibility of reducing the dimension of the model.

Due to the standard (linear) agening structure in our model (2.3) and (2.4), we may express the size of age class i at time t as a function of the number of newborn in year $t-i+1$, i.e.,

$$M_i(t) = (\prod_{k=1}^{i-1} \sigma_k^M) M_1(t-i+1), \tag{4.2}$$

and

$$F_j(t) = (\prod_{l=1}^{j-1} \sigma_l^F) F_1(t-j+1), \tag{4.3}$$

where $2 \le i, j \le n$. Hence, we may express the number of newborn as follows:

$$(M_1(t+1), F_1(t+1)) = (\nu \sum_{i,j} \phi_{ij} X_{ij}(M(t), F(t)), \sum_{i,j} \phi_{ij} X_{ij}(M(t), F(t))). \tag{4.4}$$

Thus, by using (4.2), (4.3) and (4.4), we may reduce our original model to a (lagged) two-dimensional model. If this model possesses a fixed or periodic point, then this must be the case for the original model too. We can even continue one step further: Since the sex ratio at birth, ν, is assumed to be constant, a fixed point of the above two-dimensional model must be on the form $(\nu x, x)$. Hence we get a one-dimensional version of the model defined by the map:

$$h(x) = \sum_{i,j} \phi_{ij} X_{ij}(\nu x, \nu\sigma_1^M x, ..., \nu(\prod_{k=1}^{n-1} \sigma_k^M) x, x, \sigma_1^F x, ..., (\prod_{l=1}^{n-1} \sigma_l^F) x), \tag{4.5}$$

for $x \geq 0$. One-dimensional models are simple from a computational point of view, and are easy to analyse by means of graphical techniques. Much of the dynamics of the original system may be deduced from the dynamics of the corresponding one-dimensional system. Thus, the equilibria of the above one-dimensional model may be detected by pure graphical analysis. They are all represented by the intersections between the graph of $h(x)$ and the line $y = x$. Specifically, consider the marriage function (3.1), and the expressions (3.2) and (3.3). At an equilibrium $\tilde{x}$ of the one-dimensional map (4.5) (satisfying $h(\tilde{x}) = \tilde{x}$), we have

$$F_j = (\prod_k^{j-1} \sigma_k^F)\tilde{x}, \ M_i = (\prod_k^{i-1} \sigma_k^M)\nu\tilde{x}$$

whenever $2 \leq i, j \leq n$, and $F_1 = \tilde{x}$, $M_1 = \nu\tilde{x}$. Thus we have

$$X_{ij} = \frac{\nu c_{ij}(\prod_{k=1}^{j-1} \sigma_k^F)(\prod_{k=1}^{i-1} \sigma_k^M)\tilde{x}^2}{A_i B_j} \tag{4.6}$$

where

$$A_i = 1 + \tilde{x} \sum_k \frac{\alpha_{ik}}{B_k} \tag{4.7}$$

and

$$B_j = 1 + \nu\tilde{x} \sum_k \frac{\beta_{kj}}{A_k} \tag{4.8}$$

for $i, j = 1, ..., n$, where

$$\alpha_{ik} = c_{ik}(\prod_{l=1}^{k-1} \sigma_l^F)$$

and

$$\beta_{kj} = c_{kj}(\prod_{l=1}^{k-1} \sigma_l^M).$$

When $\tilde{x}$ large ($\gg 1$), we may find real constants $\tilde{\alpha}_i$, $\tilde{\beta}_j$ and r, $0 < r < 1$ such that $A_i \approx \tilde{\alpha}_i \tilde{x}^r$ and $B_j \approx \tilde{\beta}_j \nu \tilde{x}^{1-r}$. This may be verified by inserting the above expressions into the equations (4.7) and (4.8), and by the fact that this system of equations possess a uniquely determined solution. Thus, in this case the mating function X_{ij} (4.6) is approximately equal to

$$X_{ij} \approx \frac{\nu c_{ij}(\prod_{k=1}^{j-1} \sigma_k^F)(\prod_{k=1}^{i-1} \sigma_k^M)\tilde{x}^2}{\tilde{\alpha}_i \tilde{x}^r \tilde{\beta}_j \nu \tilde{x}^{1-r}} = \frac{c_{ij}(\prod_{k=1}^{j-1} \sigma_k^F)(\prod_{k=1}^{i-1} \sigma_k^M)}{\tilde{\alpha}_i \tilde{\beta}_j} \cdot \tilde{x} \tag{4.9}$$

making the model almost linear (remember that $h(x)$ is a linear combination of the X_{ij}).

This is confirmed by numerical simulations, which indicates that the graph of $h(x)$ becomes asymtotic linear when x increases. The main weakness of the above strategy is that the original higher-dimensional system may have fixed or periodic points which is not possible to detect by examining the one-dimensional system, making the above analysis incomplete.

4.1 Results from numerical simulations

We have carried out a number of numerical simulations, with different number of age classes, and different parameters (we have only varied the birth and death rates). First we simulated a 20-dimensional model with 10 age groups of each sex. Secondly, we reduced the number of categories to four of each sex, yielding an 8-dimensional model. In both cases, we have reduced the dimension of the model to one as described above. Based on a number of simulation experiments, the following pattern seems to emerge:

(i) The map $h(x)$ given by (4.5) is almost linear.

(ii) $h(x)$ is monotonic increasing (more newborn lead to more adults).

(iii) $h(0) = 0$, that is, 0 is a (stable) equilibrium.

(iv) There are parameter values (birth/death rates and preference parameters) for which h does not possess non-trivial equilibria.

(v) $h(x)$ is (almost) convex. More precisely, there is a $K > 0$ such that $h(x)$ is convex for all $x \in (0, K)$. On the other hand, given $K > 0$, one can always find parameters such that $h(x)$ is convex for all $x \in (0, K)$.

(vi) There are parameter values for which h possesses (at least) one non-trivial equilibrium. Since 0 is a stable equilibrium, and $h(x)$ is convex, the smallest non-trivial equilibria has to be unstable. As a consequence of the former observation, the smallest non-trivial equilibrium x_0 must satisfy $x_0 < K$ if it exists.

(vii) For some parameter values, there appears to be a set of x values such that $h(x)$ is concave. However, this is apparently not enough to guarantee a new intersection with the line $y = x$, yielding a new non-trivial *stable* equilibrium. A more thorough simulation experiment is needed to settle this question.

(viii) Because of the almost linearity of h, the graph of h almost follow the line $y = x$ for some parameter values.

4.2 Discussion

From the above analysis, we may conclude that a stable, non-trivial equilibrium of our population model does not seem to exist in the case where there are no transaction costs associated with the dissolution of marriages. Hence, according to our model, the population will either continue to grow until it reaches its biological carrying capacity, when a collapse may occur (Malthus' principle), or (slowly) decrease until it becomes extinct. Since h is almost linear, the growth of the population will also be almost linear, in accordance with classical models. Thus, our analysis demonstrates that the case of a constant growing population is not merely the result of a pure 'mathematical' construction, but may be a consequence of the behavior of men and women in the marriage market.

The above analysis indicates that spectacular dynamical phenomena as cycles and chaos does not occur in our model. The only type of bifurcation we have observed in the numerical simulations is the birth of an unstable fixed point. However, we must emphasize that our analysis is based on a drastical simplification of actual realistic patterns.

Appendix A: Inclusion of divorce rates

So far we have assumed the so-called *Southern California life style*, in which a marriage can costlessly be dissolved after one year. This seems unrealistic in most populations, since the cost associated with a divorce may often be rather high. One way to account for this in the model is to introduce transaction costs into the model. This would induce 'state dependence' in the model.

One way to relax the assumtion of costless dissolutions of marriages is to assume that marriage dissolutions occur with some probability ρ. We shall now outline this approach. To this end let $Y_{ij}^M(t)$ denote the population of men of age i married to women of age j at time t and $\tilde{M}_i(t)$ the population of single men of age i in year t. Then

$$M_i(t) = \tilde{M}_i(t) + \sum_j Y_{ij}^M(t) \tag{4.1}$$

is the total number of men of age i at time t. The number of single men of age $i+1$ at time $t+1$ is equal to the number of survived single men of age i at time t who do not marry in $(t, t+1]$ plus the number of survived married men of age i at time t who divorce. Thus, if we define ρ_{ij} to be the rate of divorce between men of age i and women of age j, we have

$$\tilde{M}_{i+1}(t+1) = \sigma_i^M \cdot [\tilde{M}_i(t) - \sum_j (X_{ij}(\tilde{M}(t), \tilde{F}(t)) - \rho_{ij} \cdot Y_{ij}^M(t))] \tag{4.2}$$

where, as above, X_{ij} denotes the number of marriages (during one year), $\tilde{F}(t)$ is the available women to the single men $\tilde{M}(t)$, and where as before σ_i^M is the survival rate of women of age i. The sum is taken over all age-classes of women. The number of men of age $i+1$ married to women of age $j+1$ at time $t+1$ is equal to the (survived) number of matches (new marriages) between (single) men of age i and women of age j at time t plus the survived number of marriages between men of age i and women of age j at time t who are not been divorced during $(t, t+1]$. In mathematical terms this yields:

$$Y_{i+1,j+1}^M(t+1) = \sigma_i^M \cdot [X_{ij}(\tilde{M}(t), \tilde{F}(t)) + (1 - \rho_{ij}) \cdot Y_{ij}^M(t)]. \tag{4.3}$$

To check that internal consistency holds, (4.1), (4.2) and (4.3) yields:

$$\begin{aligned}
M_{i+1}(t+1) &= \tilde{M}_{i+1}(t+1) + \sum_j Y^M_{i+1,j+1}(t+1) \\
&= \sigma_i^M[\tilde{M}_i(t) - \sum_j (X_{ij}(\tilde{M}(t), \tilde{F}(t)) + \rho_{ij} \cdot Y^M_{ij}(t))] \\
&+ \sum_j \sigma_i^M \cdot [X_{ij}(\tilde{M}(t), \tilde{F}(t)) + (1-\rho_{ij}) \cdot Y^M_{ij}(t)] \\
&= \sigma_i^M[\tilde{M}(t) + \sum_j Y^M_{ij}(t)] \\
&= \sigma_i^M M_i(t)
\end{aligned}$$

which is as required. Analogous to the above expressions we also have formulas for updating the female population:

$$\tilde{F}_{j+1}(t+1) = \sigma_j^F \cdot [\tilde{F}_j(t) - \sum_i (X_{ij}(\tilde{M}(t), \tilde{F}(t)) + \rho_{ij} \cdot Y^F_{ji}(t))]$$

and

$$Y^F_{j+1,i+1}(t+1) = \sigma_j^F \cdot [X_{ij}(\tilde{M}(t), \tilde{F}(t)) + (1-\rho_{ij}) \cdot Y^F_{ji}(t)].$$

The above expressions may be used to define a modified demographic model, in which the mating function is equal to

$$\mu_{ij}(M(t), F(t)) = Y^M_{ij}(t) = Y^F_{ji}(t).$$

In this case, the matching game simulates the process on the marriage market between *single* males and females during one year. Applying the matching model in this way clearly provides a more intuitive and better description of reality. On the other hand, the model becomes slightly more complicated. Notice that the original approach represents the special case of the above situation, in which $\rho_{ij} = 1$ for all combinations i, j.

Since each category is supposed to represent a one-year age class, the dimension of the model may become very high. Thus, in practice, it may be a fruitful strategy to reduce the dimension by assuming that each category represents several one-year age groups. However, it is not obvious how to adjust the above formulas to cope with this situation. In this case some of the individuals will remain in the same category, while others will not. One possible way to treat this problem is to initially assume that all the individuals remain in the same category (and thus adjust the above formulas according to this). Thereafter we use the agening rates between age groups to compute how many (single, married and total) that should be moved to the next category. In the next section we present another way to treat this problem.

Appendix B: Inclusion of agening rates

In this section we shall consider an alternative way to reduce the number of age groups in the demographic model presented in the former section. Let α_i^M, α_j^F be the *agening rates* of males in age group i and females in age group j respectively, that is, the probability that a male of category i (or female of category j) at a given time t will remain in age class i (j) at time $t+1$ (for simplicity reasons we suppose these rates to be constant in time). To simplify our notation we also define

$$X_{ij}(t) \equiv X_{ij}(\tilde{M}(t), \tilde{F}(t)).$$

Then the agening of the total male population (in age group i) may be expressed as (we have a similar expression for the female population):

$$M_i(t+1) = \alpha_i^M \sigma_i^M M_i(t) + (1 - \alpha_{i-1}^M)\sigma_{i-1}^M M_{i-1}(t) \tag{4.1}$$

where $i > 1$. For age class one we must include the number of newborn (which may be written as a linear combination of the number of marriages):

$$M_1(t+1) = \alpha_1^M \sigma_1^M M_1(t) + \nu \sum_{k,l} \phi_{kl} Y_{kl}^M. \tag{4.2}$$

The population of single males in age group i at time $t+1$ will now be equal to the single males in age group i at time t not getting married who still are in age class i at time $t+1$ plus the divorced males in age group i at time t still being in category i plus the single males in age group $i-1$ at time t not getting married and becoming a member of age class i at time $t+1$ plus the divorced males in age group $i-1$ at time t being agened to category i at time $t+1$. This yields the following updating rule:

$$\begin{aligned} \tilde{M}_i(t+1) &= \alpha_i^M \sigma_i^M [\tilde{M}_i(t) - \sum_j (X_{ij}(t) - \rho_{ij} Y_{ij}^M(t))] \\ &+ (1 - \alpha_{i-1}^M)\sigma_{i-1}^M [\tilde{M}_{i-1}(t) - \sum_j (X_{i-1,j}(t) - \rho_{i-1,j} Y_{i-1,j}^M(t))]. \end{aligned} \tag{4.3}$$

Again, in the special case $i = 1$, we must remove the entries involving age class $i-1$, and include the number of newborn (males). Of course, all newborn are supposed to be single; hence

$$\tilde{M}_1(t+1) = \alpha_1^M \sigma_1^M [\tilde{M}_1(t) - \sum_j (X_{1,j}(t) - \rho_{1,j} Y_{1,j}^M(t))] + \nu \sum_{k,l} \phi_{kl} Y_{kl}^M(t). \tag{4.4}$$

To compute the number of married males of category i with females of category j at time $t+1$ we must include four cases: Marriages between males in age group i and females in age group j at time t still being in the same category at time $t+1$, marriages between males in age group $i-1$ and females in age group j where the males are agened, marriages between males of age i and females of category $j-1$ where the females are agened, and finally marriages between males in age group $i-1$ and females in age group $j-1$ where both are agened. This yields the following formula:

$$\begin{aligned} Y_{i,j}^M(t+1) &= \alpha_i^M \alpha_j^F \sigma_i^M [X_{ij}(t) + (1-\rho_{ij}) Y_{ij}^M(t)] \\ &+ \alpha_i^M (1-\alpha_{j-1}^F) \sigma_i^M [X_{i,j-1}(t) + (1-\rho_{i,j-1}) Y_{i,j-1}^M(t)] \\ &+ (1-\alpha_{i-1}^M) \alpha_j^F \sigma_{i-1}^M [X_{i-1,j}(t) + (1-\rho_{i-1,j}) Y_{i-1,j}^M(t)] \\ &+ (1-\alpha_{i-1}^M)(1-\alpha_{j-1}^F) \sigma_{i-1}^M [X_{i-1,j-1}(t) + (1-\rho_{i-1,j-1}) Y_{i-1,j-1}^M(t)] \end{aligned} \tag{4.5}$$

where $i, j > 1$. In the special case where i or $j = 1$, we must as before remove the entries indexed by $i-1$ or $j-1$ respectively:

Special case I, $i = 1, j > 1$:

$$\begin{aligned} Y_{1,j}^M(t+1) &= \alpha_1^M \alpha_j^F \sigma_1^M [X_{1,j}(t) + (1-\rho_{1,j}) Y_{1,j}^M(t)] \\ &+ \alpha_1^M (1-\alpha_{j-1}^F) \sigma_1^M [X_{1,j-1}(t) + (1-\rho_{1,j-1}) Y_{1,j-1}^M(t)]. \end{aligned} \tag{4.6}$$

Special case II, $i > 1, j = 1$:

$$\begin{aligned} Y_{i,1}^M(t+1) &= \alpha_i^M \alpha_1^F \sigma_i^M [X_{i,1}(t) + (1-\rho_{i,1}) Y_{i,1}^M(t)] \\ &+ (1-\alpha_{i-1}^M) \alpha_1^F \sigma_{i-1}^M [X_{i-1,1}(t) + (1-\rho_{i-1,1}) Y_{i-1,1}^M(t)]. \end{aligned} \tag{4.7}$$

Special case III, $i = 1, j = 1$:

$$Y_{1,1}^M(t+1) = \alpha_1^M \alpha_1^F \sigma_1^M [X_{1,1}(t) + (1-\rho_{1,1}) Y_{1,1}^M(t)]. \tag{4.8}$$

Similarly to the former case, we could verify the formulas (4.3) and (4.5) and their corresponding special cases (4.4), (4.6), (4.7) and (4.8), by first summing (4.5) over all age groups j of females, and observing that a lot of the entries are cancelling out each other. Adding the result of this computation to (4.3) yields the right hand side of (4.1) (or (4.2)), the total male population in age group i at time $t+1$, as it should be. We have similar formulas as (4.3) and (4.5) for the female population as well. The above formulas may be used to define an alternative demographic model, where each category may consist of several one-year age classes. This will reduce the dimension of the model. On the other hand, this approach requires a knowledge of the ageing parameters α_i^M, α_j^F (these may be estimated using demographic data), and the inclusion of these will increase the dimension of the parameter space. We have implemented the above configuration and run a few simulations. However, the results so far have not been significantly different from the original version, indicating that the inclusion of divorces in the model does not alter the qualitative behaviour of the model to any extent.

Appendix C. Simulation Results

Small sample and robustness properties of the marriage model

Recall that (3.1) to (3.5) represents the *asymptotic* number of marriages. Hence, it is of great interest to analyze the robustness and small sample properties of the marriage model. To investigate these properties, we have implemented the marriage model using the mathematical software GAUSS from Aptech Inc on a UNIX workstation (SUN SPARC) at Statistics Norway. The choice of language may not have been optimal with respect to the speed of the simulation. However, the choice of programming language is made partly because of our own experience from programming in GAUSS, and partly because GAUSS provides the possibility of *vectorization* of the program code, making the programs more compact. We have simulated versions of the model with different assumptions about the probability distributions of the random terms of the utility functions of the women and men.

The motivation for this is that it is of great interest to find out to which extent the predictions of the model are robust with respect to alternative probability distributions of the disturbances of the utility functions, and the introduction of flexible contracts. To throw some light on these questions, we have run several simulation experiments, using the different versions of the model.

We have done three series of simulations. In series one, we have considered the simple case of only one category of both males and females, and no flexible contract. In series two we considered the case of two age groups of both males and females, and two flexible contracts. In the last simulation series, we considered only one age group of each sex, but we allowed three different contracts. In all the experiments, we ran 1000 simulations, unless otherwise denoted in the tables displaying the results (in a couple of experiments, the simulation speed was very slow, so we abrupted the simulation before 1000 simulations were completed). In every case we carried out the simulations using two different probability distributions on the disturbance of the utility functions; the *extreme value distribution* and the *log normal distribution*. Tables with simulation results are presented below. Some of the results were also reported in Dagsvik [1998].

In general, the results of the simulations show that the small sample predictions are close to the asymptotic ones. This is particularly the case when using extreme value distributed disturbances. A few of the simulations gave poor results, especially the case of a large ratio

($\gg 1$ or $\ll 1$) between the number of males and females (in a given age group) combined with the assumption of log normal distributed disturbances. It is not easy to point out a reason for this.

Table C1: Simulation series I - one category of men and women

Number in each category		Preference parameters		Number of marriages		
				Predicted	Simulated	
M	F	a	b	X	Extr.val.(st.dev.)	Normal(st.dev.)
50	50	1/7	1/7	19.273	18.05 (3.0)	16.18 (2.7)
50	150	1/7	1	31.44	30.17 (3.3)	25.49 (3.1)
60	80	1	0.5	55.48	54.56 (2.0)	54.12 (?)
30	15	1	1	14.11	13.58 (1.1)	13.72 (1.3)
30	20	0.25	1	15.64	14.67 (1.7)	14.12 (1.7)
15	20	1/6	1/8	3.79	3.32 (1.5)	3.5 (1.5)
10	15	1/9	1/3	3.07	2.57 (1.3)	2.4 (1.2)
15	90	1/20	1	11.94	10.05 (1.7)	6.54 (2.0)
20	40	1/3	1/4	13.73	12.72 (2.0)	12.56 (2.1)
10	5	0.5	1	3.78	3.32 (1.0)	3.22 (0.9)
80	40	1	0.2	35.92	33.78 (2.1)	30.23 (2.7)
30	70	1/30	1/7	6.93	6.25 (2.0)	4.56 (1.8)
20	20	1/3	1/2	11.93	10.57 (1.8)	10.41 (1.9)
8	15	1/2	1/5	4.16	3.57 (1.3)	3.51 (1.3)
90	15	1	1/20	11.94	10.16 (1.7)	6.37 (1.9)

Simulation series II - two categories - two contracts

M_i is the number of men in age group i, F_j is the number of women in age group j, the double index i, j indicates the matching between males from age group i with females of age group j. $a_{ij}(\omega)$ is the preference parameters for a man from group i to be married with a woman from age group j under the contract ω. $b_{ji}(\omega)$ is the preference parameters for a woman from group j to be married with a man from age group i under the contract ω.

Table C2: $M_1 = 20, M_2 = 15,\ F_1 = 30, F_2 = 8.$

Contract ω		1				2		
Age index i,j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	1.0	2.0	1.0	1.0	2.0	1.0	3.0	2.0
$b_{ji}(\omega)$	4.0	3.0	0.5	1.0	1.0	0.5	1.0	3.0
Predicted	7.569	0.382	3.672	2.224	11.353	0.191	3.672	4.449
Extr.val.	7.055	0.574	3.801	2.22	11.348	0.326	3.797	3.947
St.dev.	2.1	0.7	1.5	1.2	2.2	0.5	1.5	1.4
Normal	5.649	0.359	3.667	2.137	13.284	0.167	3.424	4.609
St.dev.	2.0	0.6	1.5	1.3	2.1	0.4	1.5	1.4

Table C3: $M_1 = 30, M_2 = 10,\ F_1 = 5, F_2 = 20.$

Contract ω		1				2		
Age index i,j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	1.0	2.0	1.0	1.0	2.0	1.0	3.0	2.0
$b_{ji}(\omega)$	4.0	3.0	0.5	1.0	1.0	0.5	1.0	3.0
Predicted	1.939	7.508	0.059	2.732	2.908	3.754	0.059	5.465
Extr.val.	2.194	7.437	0.065	2.548	2.578	3.679	0.086	5.487
St.dev.	1.1	1.8	0.3	1.4	1.1	1.6	0.3	1.5
Normal	2.484	7.815	0.015	2.336	2.459	2.462	0.022	6.494
St.dev.	1.1	1.7	0.1	1.4	1.1	1.4	0.2	1.6

Table C4: $M_1 = 20, M_2 = 60,\ F_1 = 10, F_2 = 30.$

Contract ω		1				2		
Age index i,j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	1.0	2.0	1.0	1.0	2.0	1.0	3.0	2.0
$b_{ji}(\omega)$	4.0	3.0	0.5	1.0	1.0	0.5	1.0	3.0
Predicted	2.653	1.248	1.655	9.34	3.98	0.624	1.655	18.679
Extr.val.	2.997	1.651	1.448	7.899	3.383	0.896	2.048	19.275
St.dev.	1.4	1.2	1.1	2.4	1.5	0.9	1.3	2.6
Normal (922s)	4.347	0.732	0.372	3.898	4.131	0.148	1.077	25.109

Table C5: $M_1 = 15, M_2 = 20,\ F_1 = 10, F_2 = 15.$

Contract ω		1				2		
Age index i, j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	1.0	2.0	1.0	1.0	2.0	1.0	3.0	2.0
$b_{ji}(\omega)$	4.0	3.0	0.5	1.0	1.0	0.5	1.0	3.0
Predicted	3.229	1.299	0.883	4.263	4.843	0.65	0.883	8.527
Extr.val.	3.426	1.556	0.87	3.656	4.373	0.776	1.073	8.585
St.dev.	1.4	1.1	0.9	1.7	1.5	0.8	0.9	1.9
Normal	4.048	1.046	0.304	2.771	4.845	0.295	0.636	10.551
St.dev.	1.5	0.9	0.5	1.5	1.6	0.5	0.7	1.7

Table C6: $M_1 = 20, M_2 = 15,\ F_1 = 30, F_2 = 8.$

Contract ω		1				2		
Age index i, j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	5.0	4.0	0.5	0.5	1.0	1.0	8.0	7.0
$b_{ji}(\omega)$	2.0	1.0	1.0	3.0	2.0	1.0	4.0	6.0
Predicted	13.992	0.084	1.718	3.31	5.569	0.042	5.154	4.345
Extr.val.	13.291	0.11	1.998	3.321	6.02	0.047	4.776	4.293
St.dev.	2.1	0.3	1.2	1.4	1.9	0.2	1.3	1.4
Normal (980s)	14.192	0.015	1.784	3.294	5.542	0.002	4.728	4.613

Table C7: $M_1 = 30, M_2 = 10,\ F_1 = 5, F_2 = 20.$

Contract ω	1				2			
Age index i, j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	5.0	4.0	0.5	0.5	1.0	1.0	8.0	7.0
$b_{ji}(\omega)$	2.0	1.0	1.0	3.0	2.0	1.0	4.0	6.0
Predicted	3.535	6.596	0.007	4.182	1.414	3.298	0.021	5.489
Extr.val.	3.234	6.536	0.03	4.183	1.571	3.34	0.073	5.347
St.dev.	1.1	1.7	0.2	1.5	1.1	1.6	0.3	1.6
Normal	3.854	7.127	0.004	4.292	1.068	2.551	0.024	5.586
St.dev.	0.9	1.5	0.1	1.6	0.9	1.4	0.2	1.6

Table C8: $M_1 = 20, M_2 = 60,\ F_1 = 10, F_2 = 30.$

Contract ω	1				2			
Age index i, j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	5.0	4.0	0.5	0.5	1.0	1.0	8.0	7.0
$b_{ji}(\omega)$	2.0	1.0	1.0	3.0	2.0	1.0	4.0	6.0
Predicted	4.531	0.201	0.906	12.836	1.812	0.1	2.717	16.848
Extr.val. (300s)	2.767	0.83	1.497	11.93	1.413	0.4	4.24	16.717
Normal (323s)	2.935	0.331	0.731	9.699	0.755	0.049	5.529	19.913

Table C9: $M_1 = 15, M_2 = 20,\ F_1 = 10, F_2 = 15.$

Contract ω	1				2			
Age index i, j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	5.0	4.0	0.5	0.5	1.0	1.0	8.0	7.0
$b_{ji}(\omega)$	2.0	1.0	1.0	3.0	2.0	1.0	4.0	6.0
Predicted	5.976	0.305	0.384	6.267	2.39	0.152	1.152	8.226
Extr.val.	4.84	0.791	0.665	5.667	2.232	0.439	1.999	7.935
St.dev.	1.4	0.8	0.8	1.9	1.3	0.6	1.1	1.9
Normal	5.585	0.464	0.419	5.263	1.638	0.116	2.17	9.1
St.dev.	1.5	0.6	0.6	1.8	1.1	0.3	1.1	1.9

Table C10: $M_1 = 20, M_2 = 15,\ F_1 = 30, F_2 = 8.$

Contract ω			1				2	
Age index i, j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	4.0	4.0	1.5	0.5	2.0	1.0	1.0	1.0
$b_{ji}(\omega)$	0.5	0.5	0.5	0.5	1.5	0.5	1.0	2.0
Predicted	8.268	2.258	6.098	1.481	8.268	0.251	3.049	2.961
Extr.val.	8.055	2.354	5.998	1.496	8.023	0.317	3.253	2.629
St.dev.	2.1	1.2	1.7	1.1	2.1	0.5	1.5	1.3
Normal	8.212	2.322	5.799	1.486	8.166	0.139	2.939	2.917
St.dev.	2.1	1.2	1.8	1.1	2.2	0.4	1.5	1.3

Table C11: $M_1 = 30, M_2 = 10,\ F_1 = 5, F_2 = 20.$

Contract ω			1				2	
Age index i, j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	4.0	4.0	1.5	0.5	2.0	1.0	1.0	1.0
$b_{ji}(\omega)$	0.5	0.5	0.5	0.5	1.5	0.5	1.0	2.0
Predicted	2.181	12.558	0.363	1.856	2.181	1.395	0.181	3.713
Extr.val.	2.013	11.841	0.451	1.846	1.99	1.828	0.306	3.698
St.dev.	1.1	1.8	0.6	1.3	1.1	1.2	0.5	1.5
Normal	2.001	12.533	0.428	1.624	1.981	0.915	0.283	4.295
St.dev.	1.1	1.6	0.6	1.1	1.0	0.9	0.5	1.5

Table C12: $M_1 = 20, M_2 = 60,\ F_1 = 10, F_2 = 30.$

Contract ω		1				2		
Age index i,j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	4.0	4.0	1.5	0.5	2.0	1.0	1.0	1.0
$b_{ji}(\omega)$	0.5	0.5	0.5	0.5	1.5	0.5	1.0	2.0
Predicted	2.225	5.456	3.622	7.894	2.225	0.606	1.811	15.788
Extr.val.	1.68	5.11	3.829	8.059	1.659	0.815	2.582	15.6
St.dev.	1.2	1.8	1.5	2.4	1.2	0.8	1.3	2.7
Normal	1.642	5.136	3.718	5.532	1.668	0.262	2.482	18.78
St.dev.	1.2	1.8	1.5	2.1	1.1	0.5	1.4	2.5

Table C13: $M_1 = 15, M_2 = 20,\ F_1 = 10, F_2 = 15.$

Contract ω		1				2		
Age index i,j	1,1	1,2	2,1	2,2	1,1	1,2	2,1	2,2
$a_{ij}(\omega)$	4.0	4.0	1.5	0.5	2.0	1.0	1.0	1.0
$b_{ji}(\omega)$	0.5	0.5	0.5	0.5	1.5	0.5	1.0	2.0
Predicted	2.988	4.416	2.44	3.205	2.988	0.491	1.22	6.411
Extr.val.	2.641	4.372	2.598	3.146	2.525	0.655	1.558	6.109
St.dev.	1.3	1.5	1.3	1.5	1.3	0.7	1.1	1.7
Normal	2.482	4.546	2.489	2.686	2.539	0.355	1.516	6.866
St.dev.	1.3	1.5	1.3	1.4	1.4	0.6	1.1	1.7

Simulation series III - one category, three flexible contracts

Table C14: $M = 30, F = 25.$

Contract ω	1	2	3
$a(\omega)$	5.0	4.0	3.0
$b(\omega)$	2.0	1.5	2.5
Predicted	10.552	6.331	7.914
Extr.val. (St.dev.)	9.679 (2.4)	6.241 (2.2)	8.658 (2.4)
Normal (St.dev.)	9.731 (2.6)	5.201 (2.1)	9.844 (2.6)

References

Caswell H. & Weeks D. [1986]. Two-Sex Models: Chaos, Extinction, and other Dynamic Consequences of Sex. *The American Naturalist* 128, 707-735.

Chung R. [1994]. Cycles in the Two-Sex Problem: An Investigation of a Non-linear Demographic Model. *Mathematical Population Studies* 5(1), 45-73.

Dagsvik J.K. [1998]. Aggregation in Matching Markets. *Discussion Paper* 173, Statistics Norway.

Dagsvik J.K., Flaatten A.S., & Brunborg H. [1998]. A Behavioral Two-sex Marriage Model. *Discussion Paper* 238, Statistics Norway.

Guckenheimer J. & Holmes P. [1983]. Non-Linear Oscillators, Dynamical Systems, and Bifurcations of Vector Fields. Springer Verlag.

Hartman P. [1964]. Ordinary Differential Equations. John Wiley & Sons.

Hirsch M.W., Pugh C. & Shub M. [1977]. Invariant Manifolds. *Lecture Notes in Mathematics* 583, Springer Verlag.

Keyfitz N. [1972]: The Mathematics of Sex and Marriage. In: *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* 4, University of California Press, Berkeley, 89-108.

Leslie P.H. [1945]. On the Use of Matrices in Certain Population Mathematics. *Biometrika* 33, 183-212.

McFarland D.D. [1972]. Comparision of Alternative Marriage Models. In Grenville T.N.E. (ed.) *Population Dynamics.* Academic Press, New York.

Pollak R.A. [1990]. Two-Sex Demographic Models. *Journal of Political Economy* 98(21).

Pollard J.H. [1995]. Modelling the Interaction between the Sexes. Acturial Studies and Demography. Research Paper, School of Economic and Financial Studies, McQurrie University, Australia.

Roth A.E. & Sotomayor M.A.O. [1990]. Two-Sided Matching. Cambridge University Press, New York.