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FOUR PAPERS ON THE ANALYTIC GRADUATION OF FERTILITY CURVES

By
Jan M. Hoem, Erling Berge and
Britta Holmbeck

FIRE ARTIKLER OM ANALYTISK GLATTING AV FRUKTBARHETSKURVER

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PREFACE

The diagram of age-specific fertility rates for a regional population will typically be so ragged that smoothing of the fertility curve is a first step needed for analysis and projection. The fitting of a parametric function to the curve (analytic graduation) is one of the smoothing methods in current use, and the Central Bureau of Statistics of Norway has previously printed contributions to its theory in Articles 49 and 51. The present article contains four further papers with theoretical and empirical results on the analytic graduation of fertility curves, in particular on graduation by means of the Hadwiger function. The Bureau is grateful to the original publishers for permission to reprint the papers. Their original paginations have been retained.

Central Bureau of Statistics, Oslo, 5 July 1976

Petter Jakob Bjerve

FORORD

Et diagram over aldersspesifikke fødselsrater for en regional befolkning vil vanligvis være så uregelmessig at glatting av fruktbarhetskurven blir et nødvendig første skritt i analyse og framskriving. Tilpasning av en parametrisk funksjon til kurven (analytisk utjevning) er en av de glattingsmetoder som kan brukes, og Statistisk Sentralbyrå har tidligere trykt bidrag til metodens teori i Artiklene 49 og 51. En utgir nå samlet ytterligere fire arbeider med teoretiske og empiriske bidrag om analytisk glatting av fruktbarhetskurver, særlig om tilpasning av Hadwigerfunksjonen. Byrået takker for tillatelse til å gjenopptrykke artiklene. En har beholdt den opprinnelige pagineringen.

Statistisk Sentralbyrå, Oslo, 5. juli 1976

Petter Jakob Bjerve

CONTENTS

THEORETICAL AND EMPIRICAL RESULTS ON THE ANALYTIC GRADUATION OF FERTILITY CURVES

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1. INTRODUCTION

1 A. The diagram of age-specific fertility rates for a population, based on data for a calendar period, say, will typically picture a curve which looks much like a left-skewed unimodal probability density, such as a gamma density or some of the beta densities (starting just below age 15), for instance, but with superposed fluctuations. Unless the population is very large, the diagram of the sequence of fertility rates, plotted against age, will have quite a ragged appearance. It is frequently assumed that "real fertility" would be portrayed by a smooth curve and that the irregularities of the observed curves are due to accidental circumstances. The observed fertility rates are then regarded as "raw" or primary estimates of the underlying "real" rates, and graduation is employed to get a smoother curve.

1 B. When the smooth curve is produced by fitting a nice, parametric function to the original data, we call it analytic graduation. The present paper reports some findings on the following question : Given that one has chosen a particular graduating function with a specified parametrization, which method should one select to fit the function to the observed fertility data?

Some previous research (Hoem, 1972) has shown that a modified minimum *chi*-square method cannot be outdone by any other weighted or unweighted least squares method or by any moment method, in the sense that asymptotically as the population size increases the modified minimum *chi*-square estimators have minimal variances within the class of estimators considered. This holds for the estimators of the parameters of the fitting function as well as for the estimators of the function values (i.e., of the "true" fertility rates at all ages). So far, one has not known, however, to what extent this theoretical result has practical consequences in that the numerical values of the asymptotic variances of *chi*-square estimators are noticeably smaller than the corresponding variance values for reasonable competitors like least squares estimators.

1 C. We have applied analytic graduation methods to a number of empirical fertility curves, and this paper reports briefly on some selected typical findings, It turns out that estimated asymptotic variances (and the corresponding estimated coefficients of variation) almost without exception are much smaller for rates graduated by least squares than for the original ungraduated rates. In every case which we have investigated, there is also some further real gain in estimated variance in going from least squares graduation to minimum chi-square graduation. On a rare occasion, the extra gain in variance is as "low" as some 10 per cent, but it frequently is higher, and it can be substantial. Thus the optimality (in terms of estimated asymptotic variances) of the modified minimum chi-square method does have considerable practical interest.

1 D. The presentation goes as follows :

In Section 2 we summarize results given in the previous paper, but restated here in a form meant to be more easily accessible. The account is largely phrased in terms of fertility graduation, and it is of course relevant to this problem, but the reader should bear in mind that the theory is by no means limited to the case of fertility. It can be applied to any type of occurrence/exposure rate.

Section 3 then contains the selected empirical results. They are based on curves of observed fertility rates, and they are relevant for such curves as well as for other curves of vital rates of a similar form, such as marriage rates, rates of migration, etc.

2. THEORETICAL RESULTS

2 A. The "raw" age-specific fertility rates in a set will have been calculated for a number of age groups, which for simplicity we shall take to be the single-year age intervals $\alpha, \alpha+1, \cdots, \beta$. Let $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_\alpha, \hat{\lambda}_{\alpha+1}, \cdots, \hat{\lambda}_\beta)'$ denote the vector of "raw" rates, the prime signifying a transpose. These rates are to be graduated by means of some sequence of parametric functions

$$\mathbf{g}(\boldsymbol{\theta}) = (g_\alpha(\boldsymbol{\theta})\, g_{\alpha+1}(\boldsymbol{\theta}), \cdots, g_\beta(\boldsymbol{\theta}))',$$

that is, one is required to select some value $\hat{\boldsymbol{\theta}}$ of the vector $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_r)$, such that $\mathbf{g}(\hat{\boldsymbol{\theta}})$ fits as well as possible to the "raw" rates in $\hat{\boldsymbol{\lambda}}$. In our particular empirical situation, $g_x(\boldsymbol{\theta})$ is the value of, say, the Hadwiger function at argument x, so that

$$g_x(\boldsymbol{\theta}) = \frac{RH}{T\sqrt{\pi}}\left(\frac{T}{x-D}\right)^{3/2} \exp\left\{-H^2\left(\frac{T}{x-D} + \frac{x-D}{T} - 2\right)\right\}.$$

Other investigations (Hoem and Berge, 1974) have shown that it pays to take

$$\theta_1 = R,\ \theta_2 = D + T\left\{\left(1 + \frac{16}{9}H^4\right)^{1/2} - 1\right\} \Big/ \left(\frac{4}{3}H^2\right),$$

$$\boldsymbol{\theta}_3 = T + D, \text{and } \theta_4 = \frac{1}{2}\ T^2/H^2$$

as the basic parameters of the graduation. (If we regard $g_x(\boldsymbol{\theta})$ as a function of a continuous x, $g_x(\boldsymbol{\theta})/R$ then becomes a probability density with mode θ_2, mean θ_3, and variance θ_4.)

We shall take this to mean that there is some underlying "true" vector $\boldsymbol{\lambda}^0 = \{\lambda^0_\alpha, \ldots, \lambda^0_\beta\}'$ of fertility rates for the age intervals in question, and that the task at hand is to remove as well as possible the random deviation $\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}^0$. For our purposes, $\boldsymbol{\lambda}^0$ can be represented with sufficient accuracy by $\mathbf{g}(\boldsymbol{\theta}^0)$, where $\boldsymbol{\theta}^0$ is a corresponding "true" value of $\boldsymbol{\theta}$. Then, $\boldsymbol{\theta}^0$ is estimated by $\hat{\boldsymbol{\theta}}$ and $\boldsymbol{\lambda}^0$ by $\mathbf{g}(\hat{\boldsymbol{\theta}})$.

2 B. The fitting can be done in various ways, but the ordinary (i.e., unweighted) least squares method and the modified minimum chi-square method are immediate and prominent candidates. They consist, of course, of minimizing

$$Q_{LS}(\boldsymbol{\theta}) = N \sum_{x=\alpha}^{\beta} \{\hat{\lambda}_x - g_x(\boldsymbol{\theta})\}^2$$

and

$$Q_{CS}(\boldsymbol{\theta}) = \sum_{x=\alpha}^{\beta} \frac{\{\hat{\lambda}_x - g_x(\boldsymbol{\theta})\}^2}{\hat{\sigma}^2_x/N} = \sum_{x=\alpha}^{\beta} \frac{\{B_x - L_x g_x(\boldsymbol{\theta})\}^2}{B_x}$$

Here, N denotes the total number of women under observation, i.e., the number of women who are potential contributors to the numbers of liveborn babies counted. B_x is the number of liveborn babies with mothers at age x at the time of childbearing, and L_x is the total number of person-years observed at age x, so that $\hat{\lambda}_x = B_x/L_x$. Furthermore, $\hat{\sigma}^2_x = N\hat{\lambda}_x/L_x$. We can regard $\hat{\sigma}^2_x/N$ as an estimator of the asymptotic variance of $\hat{\lambda}_x$. These two methods are members of a whole class of procedures which are generated in the following way:

Let $\mathbf{M}$ be a square matrix of elements m_{xy} $(x, y = \alpha, \alpha+1, \ldots, \beta)$ which may (but need not) be random variables (by depending on the data). Let

$$Q_{\mathbf{M}}(\hat{\boldsymbol{\theta}}) = N\{\hat{\boldsymbol{\lambda}} - \mathbf{g}(\boldsymbol{\theta})\}' \mathbf{M}\{\hat{\boldsymbol{\lambda}} - \mathbf{g}(\boldsymbol{\theta})\} = N \sum_x \sum_y m_{xy}\{\hat{\lambda}_x - g_x(\boldsymbol{\theta})\}\{\hat{\lambda}_y - g_y(\boldsymbol{\theta})\},$$

and assume that there exists a value of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}(\mathbf{M})$, which minimizes $Q_{\mathbf{M}}(\boldsymbol{\theta})$. Then $\mathbf{g}\{\hat{\boldsymbol{\theta}}(\mathbf{M})\}$ represents one possible graduation of $\boldsymbol{\lambda}$.

If $\mathbf{M} = \mathbf{I}$, the graduation is by means of least squares.

Let $\hat{\boldsymbol{\sigma}} = \text{diag}(\hat{\sigma}^2_\alpha, \hat{\sigma}^2_{\alpha+1}, \ldots, \hat{\sigma}^2_\beta)$. Then, if $\mathbf{M} = \hat{\boldsymbol{\sigma}}^{-1} = \text{diag}(1/\hat{\sigma}^2_\alpha, \ldots, 1/\hat{\sigma}^2_\beta)$, $\mathbf{g}\{\hat{\boldsymbol{\theta}}(\mathbf{M})\}$ represents a graduating by modified minimum chi-square.

2 C. In order to choose between the various possible methods of fitting $\mathbf{g}(\boldsymbol{\theta})$ to $\hat{\boldsymbol{\lambda}}$, we must know something about their statistical properties. A framework for a discussion of these can be found in a previous paper (Hoem, 1972), and we shall recapitulate a few of the main results stated there. These results hold under assumptions which were given in that paper, and which we shall not repeat in full here, but which we can reasonably assume to hold for the kind of practical situation which we have in mind.

We shall assume, then, that as the population size N increases, $\hat{\boldsymbol{\lambda}}$ is asymptotically multinormally distributed with mean $\boldsymbol{\lambda}^0$ and some covariance matrix $\boldsymbol{\Sigma}_0/N$. [In our empirical studies of fertility, we take $\boldsymbol{\Sigma}_0$ to be diagonal, i.e., we take the $\hat{\lambda}_x$ to be asymptotically independent.]

We shall assume also that as $N \to \infty$, $\mathbf{M}$ converges in probability to some positive definite matrix $\mathbf{M}_0$.

Then $\hat{\boldsymbol{\theta}}(\mathbf{M})$ is asymptotically multinormally distributed with mean $\boldsymbol{\theta}^0$ and a covariance matrix $\boldsymbol{\Sigma}/N$, which satisfies

$$\boldsymbol{\Sigma} = \mathbf{A}\,\boldsymbol{\Sigma}_0\,\mathbf{A}', \tag{2.1}$$

with

$$\mathbf{A} = (\mathbf{J}_0'\,\mathbf{M}_0\,\mathbf{J}_0)^{-1}\,\mathbf{J}_0'\,\mathbf{M}_0. \tag{2.2}$$

Here, $\mathbf{J}_0 = \mathbf{J}(\boldsymbol{\theta}^0)$, where

$$\mathbf{J}(\boldsymbol{\theta}) = \begin{Bmatrix} \frac{\partial}{\partial\theta_1} g_\alpha(\boldsymbol{\theta}), \ldots, \frac{\partial}{\partial\theta_r} g_\alpha(\boldsymbol{\theta}) \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \frac{\partial}{\partial\theta_1} g_\beta(\boldsymbol{\theta}), \ldots, \frac{\partial}{\partial\theta_r} g_\beta(\boldsymbol{\theta}) \end{Bmatrix}.$$

Similarly, $\mathbf{g}(\hat{\boldsymbol{\theta}}(\mathbf{M}))$ is asymptotically multinormally distributed with mean $\boldsymbol{\lambda}^0 = \mathbf{g}(\boldsymbol{\theta}^0)$ and (singular) covariance matrix $\mathbf{J}_0\,\boldsymbol{\Sigma}\,\mathbf{J}_0'/N$. If, in particular, $\mathbf{M}_0 = \boldsymbol{\Sigma}_0^{-1}$ then $\boldsymbol{\Sigma}$ is equal to

$$\boldsymbol{\Sigma}_{0,0} = (\mathbf{J}_0'\,\boldsymbol{\Sigma}_0^{-1}\,\mathbf{J}_0)^{-1}.$$

This will be the case in our practical situation if we use the modified minimum *chi*-square method. In the general case, $\boldsymbol{\Sigma}$ will equal $\boldsymbol{\Sigma}_{0,0}$ if $\mathbf{M} = \hat{\boldsymbol{\Sigma}}_0^{-1}$, where $\hat{\boldsymbol{\Sigma}}_0$ is a consistent estimator of $\boldsymbol{\Sigma}_0$.

One can prove that for any M_0, $\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_{0,0}$ will be positive semidefinite, and so will $\mathbf{J}_0\,\boldsymbol{\Sigma}\,\mathbf{J}_0' - \mathbf{J}_0\,\boldsymbol{\Sigma}_{0,0}\,\mathbf{J}_0'$. This means, among other things, that the asymptotic variance of each $\hat{\theta}_i(\mathbf{M})$ will be minimized if $\mathbf{M}$ is chosen such that $\mathbf{M}_0 = \boldsymbol{\Sigma}_0^{-1}$. Similarly, the asymptotic variance of each $g_x(\hat{\boldsymbol{\theta}}(\mathbf{M}))$ will be minimized by the same choice of $\mathbf{M}$. Thus, in our particular empirical investigation, the use of *this* criterion of optimality will imply that one should use modified minimum *chi*-square graduation or some method with the same asymptotic properties, such as the maximum likelihood method described by Hoem (1972).

3. SOME EMPIRICAL RESULTS

3 A. We have applied the theory of the previous Section to a substantial number of Norwegian fertility curves, and a comprehensive presentation of our empirical results will be given in a forthcoming Working Paper. The present Section contains a brief account of two such cases for purposes of illustration.

3 B. We have selected the fertility curve for the city of Oslo, 1968—71, as well as the curve for the same period for an aggregate of 13 communes on the Norwegian West Coast, viz., Kvitsøy, Bokn, Utsira, Austevoll, Sund, Øygarden, Austrheim, Fedje, Solund, Askvoll, Selje, Sande, and Giske. The observed fertility curves have been plotted in Figures 1 and 2, respectively. The curve for Oslo represents a low level of fertility and a rather symmetrical age pattern of fertility for present-day Norway. (The total fertility rate by the *chi*-square fit is 1.9, while the corresponding modal and mean ages at childbearing are 25.2 and 27.3, respectively.) In contrast to this, the other curve represents a high fertility level for the Norway of today (TFR = 3.5) and a particularly skew age-pattern. (The modal and mean ages at childbearing by the chi-square fit are 24.1 and 28.4, respectively). These curves thus correspond to quite different patterns of fertility.

3 C. For each set of data, the "raw" fertility rates $\hat{\lambda}_x$ have been calculated for females for single-year age intervals, counting live offspring of both sexes. Age x corresponds to age as of December 31 in any observational year, and the contributions to L_x of any year is the arithmetic mean of the number of x-year olds at the beginning and end of the year, age being calculated as of Dec. 31 of *that* year.

The Hadwiger function of Subsection 2 A, with the parametrization $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_4)$ given there, has been fitted to the rates for ages 16 to 44. Then θ_1 corresponds to the total fertility rate, θ_2 to the modal age of childbearing, θ_3 to the corresponding mean age, and θ_4 to the variance of the age-pattern of fertility.

We have made graduations based on the method of least squares as well as separate graduations of the same data based on the modified minimum chi-square method. The graduated rates have been plotted in both Figures. Eyeball inspection judges the fit to be acceptable. As compared with the fit to other curves we have investigated, that of the two examples are about average.

The numerical calculation have been carried out on a Honeywell-Bull H6060 computer by means of a program developed by Berge (1974) on the basis of various algorithms previously published by others.

3 D. Let us denote the min.—χ^2 estimator for $\boldsymbol{\theta}$ by $\hat{\boldsymbol{\theta}}$, and the least squares estimator by $\boldsymbol{\theta}^*$. The corresponding estimates for the two sets of data mentioned have been listed in Table 1. In the case of the curve for Oslo, the estimates are pretty much the same for both methods. For the other curve, the two methods give somewhat different parameter estimates.

OBSERVED AND GRADUATED CURVES
Females, Oslo and 13 Norwegian communes 1968 — 71,

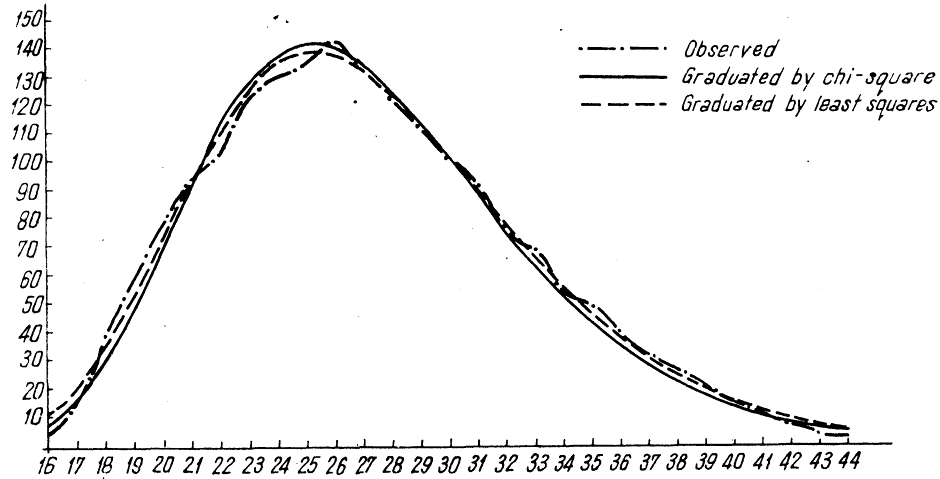

Fig. 1. Oslo Live offspring of both sexes

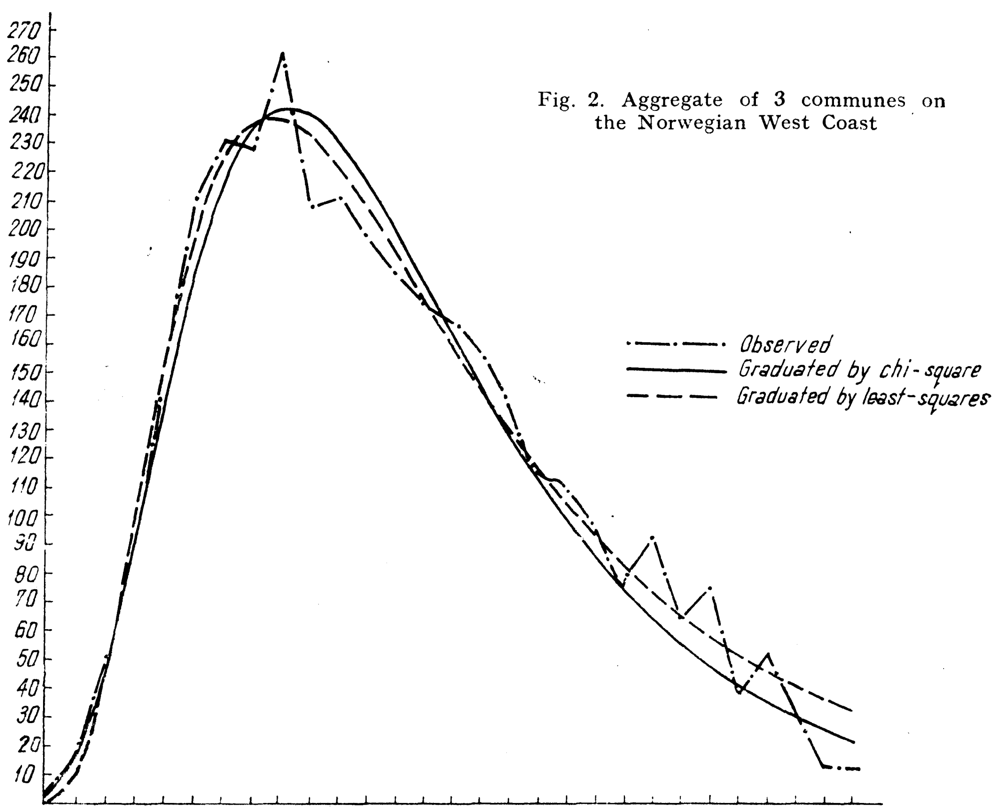

Fig. 2. Aggregate of 3 communes on the Norwegian West Coast

Table 1

Estimates of basic graduation parametres

Estimand	Min. statistic	Oslo, 1968–71		13 communes on the West Coast of Norway, 1968–71	
		By min. $-\chi^2$	By least squares	By min. $-\chi^2$	By least squares
θ_1 = R		1.88	1.93	3.50	3.72
θ_2 = mode		25.23	25.28	24.12	23.62
θ_3 = mean		27.28	27.40	28.44	29.70
θ_4 = variance		32.46	35.53	54.00	83.04
	χ^2 LSQ	186.1	$2.7 \cdot 10^{-4}$	41.9	$42.8 \cdot 10^{-4}$

Graduation of fertility curves by means of the Hadwiger function.
Single-year age intervals for ages 16–44. Rates per 1 female,counting live offspring of both sexes.

3 E. We now turn to variance estimation. We have proceeded as follows. Let $\hat{\mathbf{J}} = \mathbf{J}(\hat{\boldsymbol{\theta}})$, $\hat{\mathbf{A}} = \{\hat{\mathbf{J}}'(\hat{\boldsymbol{\sigma}}/N)^{-1}\hat{\mathbf{J}}\}^{-1}\,\hat{\mathbf{J}}'(\hat{\sigma}/N)^{-1}$, $\mathbf{A}^* = (\hat{\mathbf{J}}'\,\hat{\mathbf{J}})^{-1}\hat{\mathbf{J}}'$, $\hat{\Sigma}/N = \hat{\mathbf{A}}(\hat{\boldsymbol{\sigma}}/N)\hat{\mathbf{A}}'$, and $\boldsymbol{\Sigma}^*/N = \mathbf{A}^*\;(\hat{\boldsymbol{\sigma}}/N)\;(\hat{\mathbf{A}}^*)'$.
Then, $\hat{\boldsymbol{\Sigma}}/N$ is our estimator of the asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}$, and $\boldsymbol{\Sigma}^*/N$ is our estimator of the corresponding matrix for $\boldsymbol{\theta}^*$, on the assumption that the true age-specific fertility rates are represented adequately by the Hadwiger curve. Similarly,

$$\hat{\mathbf{J}}(\hat{\boldsymbol{\Sigma}}/N)\;\hat{\mathbf{J}}' \text{ and } \hat{\mathbf{J}}\;(\boldsymbol{\Sigma}^*/N)\hat{\mathbf{J}}'$$

are our estimators of the asymptotic covariance matrices of $\mathbf{g}(\hat{\boldsymbol{\theta}})$ and $\mathbf{g}(\boldsymbol{\theta}^*)$, respectively.

Estimates of the asymptotic variances of the $\hat{\theta}_i$ and θ_i^* have been computed and listed in Table 2. The *chi*-square method seems impressively much better than the method of least squares for these estimands.

Table 2

Estimated asymptotic variances of estimators of basic graduation parameters[1)]

Parameter	Oslo, 1968–71		13 Norwegian communes, 1968–71	
	Min.$-\chi^2$ graduation[2)] (1)	Least squares graduation (NB : In per cent of (1).) (2)	Min. $-\chi^2$ graduation[2)] (3)	Least squares graduation (NB : In per cent of (3).) (4)
θ_1 = R	142	117	6 283	122
θ_2 = mode	3 609	132	44 281	166
θ_3 = mean	1 720	186	50 255	182
θ_4 = variance	201 572	216	17 999 297	201
ΣL^*	369 178.0		18 900.5	

1) Corresponding to min. $-\chi^2$ estimates of Table 1.
2) Multiplied by 10^6.

Corresponding variance estimates for graduated age-specific fertility rates for selected ages have been listed in Table 3, along with estimates for coefficients of variation. Column 4 shows that *chi*-square graduation of

Table 3

Estimated asymptotic variances and coefficients of variation[1]) of estimators of fertility rates for some selected ages

Age as of Dec. 31	Variances				Coefficients of variation			
	$\frac{\hat{\sigma}_x^2}{N} = \frac{\hat{\lambda}_x}{L_x}$ (1)[2]	Min.-χ^2 graduation (2)[2])	Least squares graduation (NB: In per cent of (2).) (3)[3])	(1) in per cent of (2) (4)[3])	$\frac{\hat{\sigma}_x/\sqrt{N}}{\hat{\lambda}_x}$ (5)[4])	Min.-χ^2 graduation (6)[4])	Least squares graduation (NB: In per cent of (6).) (7)[3])	(5) in per cent of (6) (8)[3])
	a. Oslo, 1968 — 71							
16	0.360	0.186	224	194	16.01	5.87	150	273
18	3.144	0.454	175	692	4.63	2.20	132	210
20	4.836	0.795	112	608	2.83	1.26	106	224
24	6.424	1.053	112	610	1.95	0.75	106	261
28	9.775	1.198	115	816	2.56	0.88	107	292
35	5.572	0.325	152	1713	4.77	1.33	123	358
40	1.425	0.117	225	1215	7.88	2.50	150	315
44	0.202	0.040	238	501	18.26	4.15	154	440
	b. 13 Norwegian communes, 1968 — 71							
16	2.206	0.839	390	263	57.75	42.76	197	135
18	51.202	13.548	231	378	15.08	7.59	152	199
20	219.575	45.390	119	484	9.49	4.65	109	204
24	393.465	44.963	117	875	7.58	2.77	108	273
28	317.285	35.990	116	882	9.71	3.03	107	320
35	173.826	9.776	138	1778	13.48	3.68	118	367
40	63.956	5.960	180	1073	21.32	5.99	134	356
44	18.703	3.836	203	488	35.36	8.93	141	396

1) Corresponding to the min. $-\chi^2$ estimates of Table 1.
2) Multiplied by 10^6, i.e., corresponding to rates per 1 000 females.
3) Calculated from figures with six effective digits.
4) Multiplied by 100.

the "raw" rates can result in a substantial reduction in variance, by a factor of 2 to 18 in the cases reported here. Column 3 shows that there is some real gain in using the *chi*-square method rather than least squares for these estimands as well. For the central ages (in the twenties), the gain is some ten to twenty per cent. In the tails of the curve, the *chi*-square method is at least twice as good as least squares, as judged by the variance estimates. Given the much larger weight placed on the tail ages by the former method than by the latter, such a pattern is not surprising, of course.

The variance reduction achieved by using least squares estimates rather than the "raw" rates can be gauged by dividing each of the entries in column 4 by the corresponding entry in column 3. For most ages, the variance is reduced by a factor of 2 to 12. For age 16 in each of the two cases reported, however, the estimated asymptotic variance of the least squares estimate of the fertility rate *exceeds* that of the "raw" rate (by 15 and 48 per cent for the two curves, respectively). Such "reversals" occur occasionally at isolated ages in our data sets.

Columns 7 and 8 give similar results for the coefficient of variation.

These numerical estimates show that for this type of fertility curves, the optimality of the modified minimum chi-square method is of practical importance and is not only of theoretical interest. This conclusion is substantiated by the rest of our experience with fertility curves.

3 F. The variance estimators introduced above are functions of the chi-square estimator $\hat{\boldsymbol{\theta}}$. We have computed similar variance estimates based on the least squares estimator $\boldsymbol{\theta}^*$. The numerical results do not seem to contain much beyond what we report here.

3 G. The modified minimum chi-square value for the curve of the 13 communes is 41.9, which corresponds to the upper 1.8 percentage point of the *chi*-square distribution with 25 $(=29-4)$ degrees of freedom. Most of the chi-square values which we have calculated for curves of regional populations in Norway are better than this. For the larger communities, the *chi*-square value is invariably big, however, and our curve for Oslo is a case in point. (See line 5 of Table 1). We take this as an indication of systematic deviations between the observed and the graduated curves. This corresponds to earlier findings by others in attempts at analytic graduation of *mortality* rates.

Sometimes, deviations between the observed and the graduated curve are regarded as caused at least in part by particular historical events of some substantive interest, or as an effect of cohort differences. In such a case, the graduated curve can still be useful in providing a "standard" with which the observed curve is compared to bring out systematic deviations more clearly. We regard this as an important aspect of fertility graduation (and a potentially more realistic one than the straightforward superposed random fluctuations interpretation), and we plan to discuss it in a future communication.

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Some Problems in Hadwiger Fertility Graduation

By Jan M. Hoem and Erling Berge

Abstract

The Hadwiger function has been recommended for the analytic graduation of fertility curves. In its original parametrization, this graduating function has some rather problematic aspects in terms of parameter identifiability and interpretability. This can be remedied by a suitable reparametrization. However, a second practical identification problem arises insofar as the curves of the Hadwiger density and the gamma density with similar parameter values turn out to be virtually interchangeable in a number of numerical examples. Our impression is that the Hadwiger density is a bit more flexible than the gamma density. In this respect, therefore, the Hadwiger density seems to have a slight edge on the gamma density. These are the main concerns of this paper. In addition some findings are presented on methods of parameter estimation in current use. Moment type estimators turn out to be seriously inconsistent in several actual cases, and their replacement by minimum chi-square estimators is recommended.

1. Introduction

1 A. The diagram of age-specific fertility rates for a population, based on data for a calendar period, say, will typically picture a curve which looks much like a left-skewed probability density, but with superposed fluctuations. A number of functions have been suggested as a basis for the analytic graduation of such a diagram, the gamma and beta densities being particularly popular. Another possibility is the Hadwiger density. Empirical studies, both by ourselves and by others, have shown this to be a flexible function which gives a good fit to data from a wide range of populations. This paper presents some findings on the use of the latter density as a graduating function. It is part of the documentation of a larger project. Further empirical and theoretical results have been presented elsewhere (Hoem et al., 1974; Berge & Hoem, 1974).

1 B. Let us briefly sketch the practical background of the present study before we go into its details. The authors have investigated regional fertility in Norway around 1970. The 449 communes of the country as of January 1, 1971, have been grouped into 77 "fertility regions", mainly on the basis of their industrial composition, their geographic location, and whether they are characterized by systematic in- or out-migration. Each region will typically have a total resident

population of some 30 to 50 thousand inhabitants. For each region, age-specific fertility rates have been calculated from data for the period of 1968–1971, and the Hadwiger density has been fitted to the corresponding fertility curve.

By means of the analytic graduation, the description of regional age-specific fertility can be reduced to stating the value of a low-dimensional parameter vector for each region. Our hope is that subsequently these regional parameter vector values can be related to the values of ecological variables describing various aspects of the corresponding communities.

1 C. We have found that the four-parameter version of the Hadwiger function suggested independently by Yntema (1969) and Gilje (1969), and described again in Section 2 below, gives a very good fit to all of our 77 fertility curves, as judged by such criteria as least squares and eye-ball inspection of curve plots. This confirms similar findings by Yntema and Gilje.

[For references to the literature on fertility graduation up to about 1970, see Subsection 1.2 in Hoem (1972). Other contributions have been given by Brass (1968), Hunyadi & Szakolczai (1970), Mitra (1970), Gilje & Yntema (1971), Murphy & Nagnur (1972), Farid (1973), Romaniuk (1973), Mitra & Romaniuk (1973), Norman & Gilje (1973), and Hyrenius et al. (1974).]

1 D. During our work with this part of the project, we have discovered some problematic aspects of the use of the Hadwiger density for fertility graduation. One of us (Berge) has constructed cases where widely different values of the four original Hadwiger function parameters produce curves which are so close to each other as to be practically indistinguishable. Demographic theory does not provide a basis sufficient for choosing among the cases which have been uncovered in the graduation of real data, to say nothing about hedging against parameter value combinations which we do not know about. This means that although we are reasonably satisfied with the way in which the Hadwiger *curves* represent the level and age-pattern of fertility, we know of no way to "invert" this representation and get an interpretable and tolerably unique set of values for the Yntema–Gilje *parameters* of the Hadwiger function.

1 E. To overcome this difficulty, we have tried one of the obvious routes open to us, viz., we have sought a different parametrization of the Hadwiger function. We have taken the position that what is important, is a good fit of the curves, and we have looked for a set of four stable parameters, i.e., parameters whose values will change appreciably only when the Hadwiger curve changes in some important way. (It has not appeared possible to get a generally satisfactory curve fit with less than four parameters.) At the same time, we have wanted parameters which could be given natural demographic interpretations.

It has seemed to us that descriptive measures which are standard in any characterization of a skew bell-shaped curve, would fit our bill. For our parameters, therefore, we have selected the mode, the mean, and the variance of the Hadwiger density, as well as the area under the fertility curve. This is not such a sensational choice, and *post hoc* one may perhaps feel that we could have discovered these parameters without going through the mechanics of analytic graduation. This detour has served three purposes, however. First, it has assured us that we would need four parameters, and not some other number; and it has convinced us that these four ones are sufficient for our purposes. Secondly, the introduction of the values of these parameters into the graduating function provides us with a complete schedule of fertility rates for all ages, a fact which is useful in many circumstances, such as in connection with population projections. [Mitra & Romaniuk (1973) and Romaniuk (1973) stress the latter point.] Thirdly, analytic graduation furnishes a better set of parameter estimators than the naive ones as soon as the graduating function has been specified.

1 F. The latter point is not without its own problems, however. For comparison, we have used the gamma density as a basis for a second analytic graduation of a number of the Norwegian regional fertility curves mentioned above. The fit is roughly as satisfactory as the one given by the Hadwiger function, though the latter may possibly be a bit more flexible. This replicates earlier findings by others. We have also fitted a Hadwiger function to the graph of a gamma density, and vice versa. In all cases investigated, the Hadwiger function fits the gamma curve nicely. Similarly, the gamma function fits the Hadwiger curve well too. In both cases, the fit is occasionally somewhat less than excellent. The gamma density and the Hadwiger density seem practically interchangeable as graduating functions, at least in the cases we have studied. Thus, we are up against a second identification problem, viz., in our choice of a graduating function.

1 G. Most previous investigations have used moment type methods to fit functions to the observed fertility curves, and have not applied techniques like least squares or minimum chi-square. [Gilje (1969), Gilje & Yntema (1971), and Norman & Gilje (1974) are notable exceptions.] For the case of Hadwiger graduation, parameter estimators based on moments, the mode, etc., have been suggested by Yntema (1969), and some of their statistical properties have been discussed by Hoem (1972, Subsection 7.4).

Moment type estimators frequently have the nice feature that they can be represented by relatively simple mathematical formulas. This is not so with procedures like least squares, which will usually involve some iterative numerical method of function minimization (except in special cases, like when the graduating function is linear in its parameters). Such aspects may be part of the explanation why the more involved methods have been less popular. In our computerized age, however, the numerical work involved should be much less of a problem than it was previously, and we feel that there are good reasons, both intuitive and theoretical, to prefer minimum chi-square techniques, say, to moment methods. For one thing, a minimum chi-square or least squares criterion will frequently be used to measure the goodness-of-fit even of a graduation based on a moment method. For another, Hoem (1972) has proved that minimum chi-square methods provide estimators which are consistent and have uniformly minimum variance, and he has established the optimality of these estimators on other criteria as well (Hoem et al., 1974). Furthermore, he has pointed out (Hoem, 1972, page 591, Remark 8) that Yntema's estimators for the parameters of the Hadwiger function may be inconsistent, i.e., they may not converge in probability to the parameters estimated when the population size increases. Our numerical findings reported below show that such inconsistency can be considerable. This problem is inherent in the moment method approach to graduation, so that it is not particular to the Yntema estimators, nor is it particular to Hadwiger graduation for that matter. Indeed, we replicate this finding with tree sets of parameter values for the gamma density. In our opinion, this is a serious argument against relying solely upon such estimators.

Moment methods do have some practical interest, however, in that they can provide the starting values which are necessary for any iterative estimation procedure. The starting values can be important for the speed of convergence and other aspects of the iterative procedure, so that the statistical properties of the estimators which produce the starting values are by no means unimportant. In our experience, estimators such as those suggested by Yntema are quite sufficiently accurate for providing starting values, and we have been unable to improve upon them. Also, the problem of inconsistency seems to be less important for stable parameters than for others.

1 H. It may be of some interest to learn that we have found O'Neill's implementation of Nelder & Mead's simplex algorithm for function minimization (Nelder & Mead, 1965; O'Neill, 1971) better suited to our purposes of graduation than the Fletcher–Powell algorithm implemented by Gruvaeus & Jöreskog (1970).

2. The Hadwiger function

2 A. Let

$$f_H(x) = (H/\sqrt{\pi})\, x^{-3/2} \exp\{-H^2(x + x^{-1} - 2)\} \quad \text{for } x > 0. \tag{2.1}$$

Then f_H is a probability density on the positive real axis. We shall call it the Hadwiger density, and shall call H the Hadwiger parameter. (We follow the tradition of using capital Latin letters for the parameters in Hadwiger graduation.) The corresponding cumulant generating function is

$$\phi_H(s) = \ln \int_0^\infty e^{sx} f_H(x)\, dx = 2H^2\left\{1 - \left(1 - \frac{s}{H^2}\right)^{1/2}\right\}.$$

This distribution has a mean of 1, a variance of $1/(2H^2)$, and its kth cumulant is

$$\varkappa_k = \frac{(2k-3)(2k-5)\dots 3\times 1}{(2H^2)^{k-1}} \quad \text{for } k \geqslant 2. \tag{2.2}$$

Thus, the measures of skewness and curtosis are

$$\gamma_1 = \frac{\varkappa_3}{\varkappa_2^{3/2}} = \frac{3}{\sqrt{2}H} \quad \text{and} \quad \gamma_2 = \frac{\varkappa_4}{\varkappa_2^2} = \frac{15}{2H^2}.$$

The density is unimodal, and its mode is at

$$Z = \frac{(1+16H^4/9)^{1/2}-1}{4H^2/3}. \tag{2.3}$$

We note that $Z<1$ for all H, so that the mode is always less than the mean here.

[Formula (2.2) corrects the erroneous formula (7.24) previously given by Hoem (1972) as well as a misprint in formula (7) in Yntema (1956).]

2B. Hadwiger (1940) suggested using the function

$$h_1(x) = \frac{R}{T} f_H\left(\frac{x}{T}\right) \quad \text{for } x>0,$$

to graduate age-specific fertility rates, x representing age of mother at childbearing. (This particular parametrization is actually due to Yntema, 1952.) The area under the curve of h_1 is R, which is taken to represent the gross reproduction rate or the total fertility rate, according as only girl babies or both boy and girl babies are included in the count of the liveborn. The parameter T is taken as the mean age at childbearing, and it will typically have a value somewhere between 25 and 30 in human populations. (For a discussion of the concept of the mean age at childbearing, see Hoem, 1971.)

No one seems to have succeeded in giving convincing demographic interpretation of the Hadwiger parameter H. The most useful relation it appears in, may be

$$S^2 = \frac{T^2}{2H^2}, \tag{2.4}$$

where S^2 is the variance of the density $h_1(\cdot)/R$. For all graduations known to Yntema in 1961, H had a value slightly above 3, a fact which inspired him into calling it a demometric invariant (Yntema, 1961). (Largely, this finding has been upheld in later work.)

In Fig. 1, we have plotted the function h_1 with $R=1$ and $T=27$ for a number of values of H. With the normal type of values for the parameters, the curve of h_1 will cling to the x-axis up to some age in the early teens, whence it will start to raise its head, reach its mode at an abscissa of ZT (which will be $0.92T$ for $H=3$, or 24.8 when $T=27$), and then subside again to become very small for $x>50$.

2C. Hadwiger & Ruchti (1941) and Yntema (1952, 1953, 1956) fitted this function to their satisfaction to a number of fertility curves, but Tekse (1967) discovered that it gave a very bad fit to some important age-patterns of fertility. Yntema (1969) and Gilje (1969) then independently introduced a fourth parameter, which we shall call D, so that the graduating function took the following form:

$$h(x; R, H, T, D) = \frac{R}{T} f_H\left(\frac{x-D}{T}\right) \quad \text{for } x > D. \tag{2.5}$$

Empirical studies have shown this function to be highly flexible and to give a good fit to an extensive set of fertility age-patterns. (Some of its limitations have been explored by Yntema (1969) and by Gilje & Yntema (1971).)

2D. It is not easy, however, to give the parameters in (2.5) a reasonable demographic interpretation. The area R retains its previous interpretation as a measure of the fertility level, of course, but the mean and modal ages of childbearing now become

$$U = D + T \quad \text{and} \quad M = D + TZ, \tag{2.6}$$

respectively, so that T loses its previous interpretation as the mean, and Z loses its previous interpretation as the proportion of the mean which constitutes the mode. The disturbing element is the new parameter D whose substantive meaning is problematic. Although $h(x; R, H, T, D)$ is positive for all $x > D$, D does not have such an interpretation as the lowest fertile age, say, since the value of $h(x; R, H, T, D)$ can be negligible for a considerable interval above D. Witness the description for $D = 0$ at the end of Section 2B. In Table 1, we shall even give realistic examples where D is negative.

We had difficulty in interpreting the Hadwiger parameter H in (2.4) already, and this does not become easier in (2.5). It is not a demometric invariant any more; its value can vary widely from one graduation to the next. It still satisfies (2.4) with S^2 equal to the variance of $h(\cdot; R, H, T, D)/R$.

2E. There does not seem to be any guidance to be had from the numerical values of the parameters in empirical fits either. In fact one of us (Berge) has discovered that there are combinations of values of H, T, and D which seem completely wild as compared to what we are used to seeing in empirical graduations, yet the corresponding curves seem quite reasonable as fertility curves go. Six such combinations are given in Table 1, along with corresponding values of U, M, and S^2. Figs. 2 and 3 contain plots of the corresponding curves.

These plots bring out another interesting feature too, viz., that even though curves 1, 2 and 3 have widely different values of H, T, and D, their graphs are quite close to each other. Similarly for curves 4, 5 and 6. Normed sums of squares of deviations between pairs of curves in each group have been listed in Table 2. (A discussion of the norming procedure is given in Subsection 2F

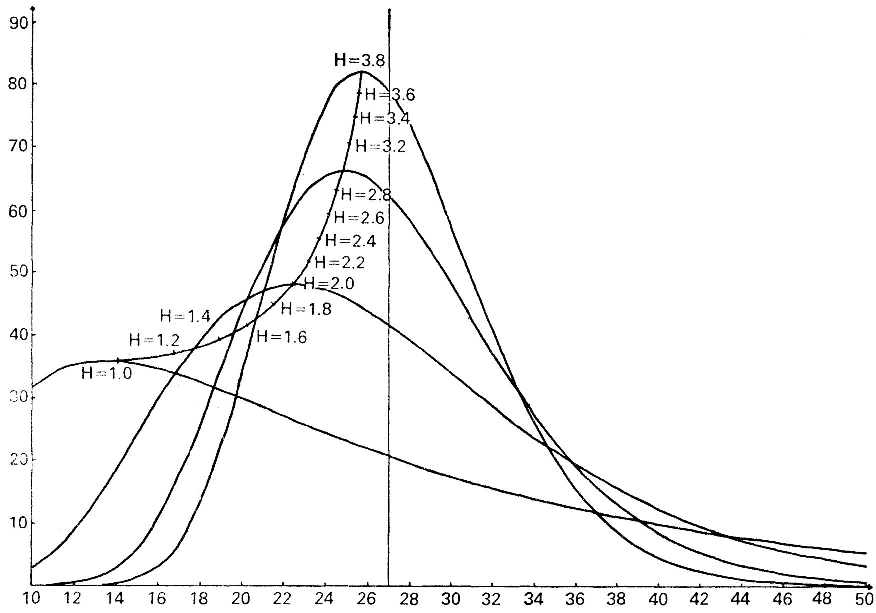

Fig. 1. Hadwiger curves and the locus of the mode corresponding to various values of H when $U=T=27$ $(D=0)$ and $R=1$.

below.) These "fits" are almost incredibly good, and we doubt whether it would be possible to choose one out of such a pair of functions if one were to graduate raw fertility rates with them.

On the other hand, the parameters R, U, M, and S^2 are quite stable (in the sense explained in Subsection 1 E) in the cases documented here as well as in a couple of other cases which we have studied. For these reasons, we recommend the replacement of the parameter vector (R, H, T, D) by (R, U, M, S^2).

2F. Each of the normed sums of squares of deviations in Table 2 has been calculated by dividing the straightforward sum of squares of deviations by the square of the mean of the corresponding two R-values. Thus, if the two curves are $R_1 f_1(x)$ and $R_2 f_2(x)$, and if $R_0 = (R_1 + R_2)/2$, then the corresponding normed sum of squares is

$$\sum_x \{R_1 f_1(x) - R_2 f_2(x)\}^2 / R_0^2.$$

The purpose of this procedure is to facilitate comparison of the fits between pairs of curves by eliminating the influence of the different areas under the various curves. Since $R_1 \simeq R_2$ in all the cases we consider, the normed sum of squares is roughly equal to

$$\sum_x \{f_1(x) - f_2(x)\}^2.$$

Table 1. *Some unconventional parameter values for the Hadwiger function which produce highly similar and reasonable fertility curves*

Curve no.	R (area)	H	T	D	$U=D+T$ (mean)	M (mode)	$S^2=T^2/(2H^2)$ (variance)
Group I							
1	0.974	5.478	48.149	− 22.107	26.042	24.854	38.628
2	0.983	14.267	125.420	− 99.766	25.654	25.193	38.640
3	0.983	51.026	443.415	−418.107	25.308	25.181	37.758
Group II							
4	1.061	51.891	515.445	−490.032	25.413	25.271	49.334
5	1.056	63.790	631.566	−605.836	25.730	25.603	49.012
6	1.049	74.961	749.431	−723.925	25.506	25.405	49.976

3. The inconsistency of moment estimators

3A. In a series of papers, Yntema (1953, 1956, 1969; Gilje & Yntema, 1971) has discussed estimators for the parameters R, H, T, and D of the Hadwiger function. One form of his estimators, as given by Hoem (1972, Subsection 7.4), is as follows.

Denote the "raw" observed fertility rate at age x by $\hat{\lambda}_x$. Let

$$\hat{R}=\sum_x \hat{\lambda}_x,\ \hat{U}=\sum_x x\hat{\lambda}_x/\hat{R},\quad \text{and } \tilde{M}=\min\{x: \hat{\lambda}_x \geqslant \hat{\lambda}_y \quad \text{for all } y\}.$$

Let $[y]$ denote the integer value of y, and let

$$\hat{V}=[\hat{U}+\tfrac{1}{2}],\ \hat{h}=\hat{\lambda}_{\hat{V}},\quad \hat{T}=\hat{R}^2/\{\tfrac{4}{3}\pi(\hat{U}-\tilde{M})\hat{h}^2\},$$

$$\hat{D}=\hat{U}-\hat{T},\quad \text{and } \hat{H}=\hat{h}\hat{T}\sqrt{\pi}/\hat{R}.$$

Then $(\hat{R}, \hat{H}, \hat{T}, \hat{D})$ are Yntema's estimators of (R, H, T, D). (Yntema, 1969, actually suggests some simple preliminary smoothing of the raw rates before $\tilde{M}$ and $\hat{h}$ are calculated.) In view of (2.3), (2.4) and (2.6), we let

$$\hat{M}=\hat{D}+\hat{T}\,\frac{(1+16\hat{H}^4/9)^{1/2}-1}{4\hat{H}^2/3} \tag{3.1}$$

Table 2. *Normed*[a] *sums of squares of deviations between pairs of curves in each group in Table 1, per 100 000*

Curve no.	Curve no. 2	3	5	6
1	3.93	9.41		
2		2.75		
4			4.07	1.46
5				2.34

[a] See Subsection 2F.

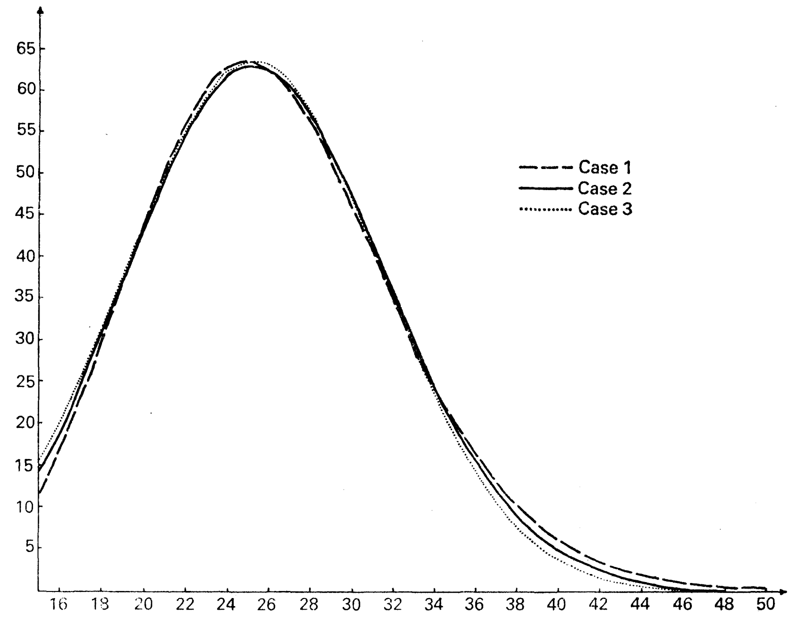

Fig. 2. Similar Hadwiger curves from different parameter sets (Table 1).

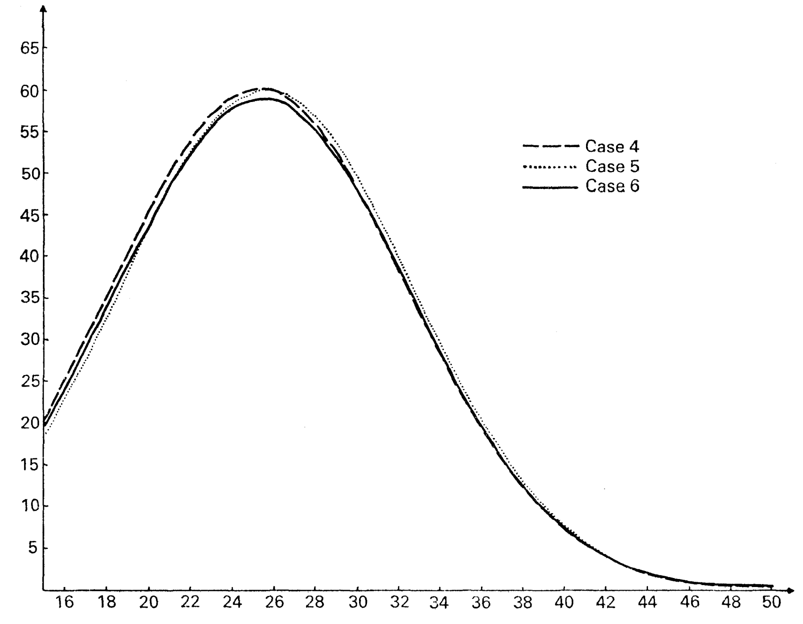

Fig. 3. Similar Hadwiger curves from different parameter sets (Table 1).

and

$$\hat{S}^2 = \frac{\hat{T}^2}{2\hat{H}^2} = \frac{\hat{R}^2}{2\pi\hat{h}^2}. \tag{3.2}$$

Following Yntema's lead, we have then used $(\hat{R}, \hat{U}, \hat{M}, \hat{S}^2)$ as moment estimators for (R, U, M, S^2). Thus, we regard $\tilde{M}$ only as a preliminary estimator for M, and substitute $\hat{M}$ for it when it has served its function.

3B. As the population size increases without bounds, $\hat{R}$, $\hat{H}$, $\hat{T}$, $\hat{D}$, $\hat{U}$, $\hat{M}$, and $\hat{S}^2$ converge in probability to values R_0, H_0, T_0, D_0, U_0, M_0, and S_0^2, respectively, and these values need not coincide with those of the parameters R, H, T, D, U, M, and S^2 which one wants to estimate. Formulas for R_0, H_0, T_0, D_0, U_0, M_0, amd S_0^2 are easily derived. Those for R_0, H_0, T_0, D_0, and U_0 are given by Hoem (1972, page 590) and will not be repeated here. The formulas for M_0 and S_0^2 are the same as (3.1) and (3.2) with $\hat{R}$, $\hat{H}$, $\hat{T}$, and $\hat{D}$ replaced by R_0, H_0, T_0, and D_0, respectively.

Previously, no one seems to have looked into the discrepancy between the R_0, H_0, etc., and the underlying parameters R, H, etc. We have listed a few comparisons in Table 3. It will be seen that we have used only two different sets of values for the underlying parameters. For both sets, we have applied Yntema's approach, with single year age groups between ages 15 and 50 (cases 1 and 2 in Table 3). In addition, we have applied his approach once more to the second set, again with single year age groups, but now with an age span from 15 to 44 (case 3).

The discrepancy between the quantities H_0, T_0, and D_0 on the one hand, and the corresponding parameters on the other hand, can be quite considerable. A comparison of cases 2 and 3 shows that even the choice of age groups can be important.

There is a corresponding discrepancy for R_0, U_0, M_0, and S_0^2, but it seems less important (at least for R_0, U_0, and M_0) than for H_0, T_0, and D_0.

3C. The formulas of Yntema's estimators have been set up by a largely intuitive argument which exploits an analogy between on the one hand the theoretical Hadwiger model, where age appears as a continuous variable, and on the other hand the real data, which are organized by age groups. The discrepancy between the probability limits of the estimators and the parameters estimated is due to the fixed discretization involved in the presentation of the real data by (say) single-year age groups. If the length of the age interval of each age group were permitted to decrease suitably to zero as the population size increases, and if the number of age groups were permitted to increase correspondingly, the discrepancy would disappear. This is not part of the kind of approach inherent in methods of analytic graduation of vital rates, however, and this type of discrepancy therefore becomes an integral part of graduation theory. It is not particular to Yntema's estimators, and we demonstrate it again in connection with the gamma density below.

Table 3. *Three sets of values of parameters R, H, T, D, U, M, and S^2 of the Hadwiger function, and the corresponding values of R_0, H_0, T_0, D_0, U_0, M_0, and S_0^2*

Note that cases 2 and 2* are based on the same data

Case no.		R	H	T	D	U	M	S^2	Single year age groups used
1	Parameter value[a]	3.716	1.206	15.440	14.260	29.700	23.669	81.967	15–50
	Yntema value[b]	3.589	1.864	21.571	7.084	28.655	24.497	66.946	
2	Parameter value[c]	2.957	1.764	18.239	9.263	27.502	23.628	53.453	15–50
	Yntema value[b]	2.925	2.293	22.549	4.666	27.215	24.228	48.336	
2*	Parameter value[c]	2.957	1.764	18.239	9.263	27.502	23.628	53.453	15–44
	Yntema value[b]	2.869	2.556	24.647	2.183	26.831	24.162	46.492	

[a] Values of *R*, *H*, *T*, etc. Source of values: Combined data for 1968–71 for 13 municipalities on the Norwegian West Coast (Kvitsøy, Bokn, Utsira, Austevoll, Sund, Øygarden, Austrheim, Fedje, Solund, Askvoll, Selje, Sande, and Giske).
[b] Values of R_0, H_0, T_0, etc. Derived by Yntema's method.
[c] Values of *R*, *H*, *T*, etc. Source of values: Gilje, 1969, p. 130 (Norway, 1966).

3D. Fortunately, the type of discrepancy demonstrated above does not seem to have important consequences for the results of analytic graduation by methods like, say, least squares. At least this is so for the numerical cases we have studied. We have proceeded as follows.

For each of the sets of values of *R*, *H*, *T*, and *D*, in cases 1 and 2 in Table 3, we have calculated the values of the Hadwiger function for ages 15, 16, ..., 50. We have then treated this set of function values, say $\lambda_{15}^*, \lambda_{16}^*, ..., \lambda_{50}^*$, as a set of "observed" fertility rates, we have calculated the corresponding Yntema "estimates", and we have finally used these as starting values for an iterative algorithm producing least squares "estimates". (Since there is nothing here that corresponds to the size of each population group, minimum chi-square "graduation" is out of the question.)

The results are impressive. The differences between the parameter values and their least squares "estimates" never exceed 1/1 000, and the sums of squares of deviations between the λ_x^* and *their* least squares estimates are of the size order of 10^{-6} of less. Such differences may well be due mostly to rounding errors.

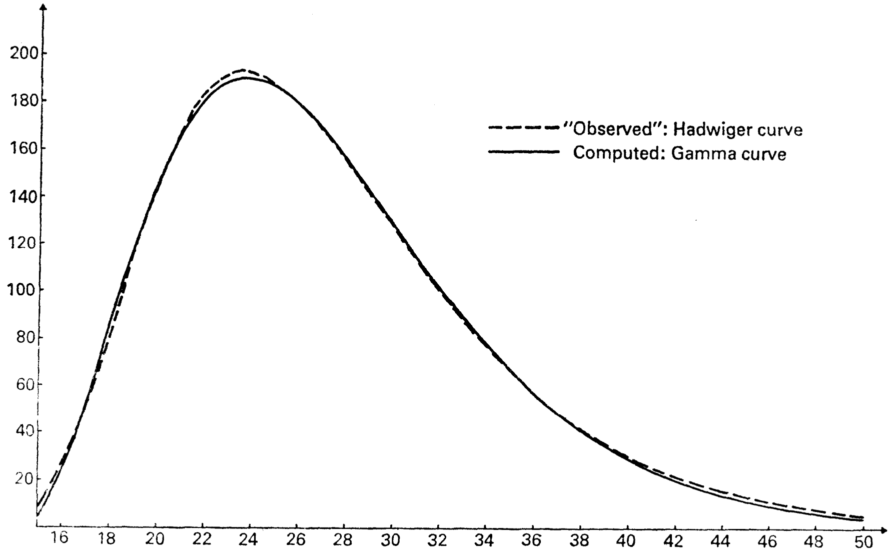

Fig. 4. Gamma graduation of Hadwiger curve for Norway, 1966 (case 2 of Table 5).

4. Graduation by means of the gamma density

4 A. For comparison with Hadwiger graduation, we shall take a brief look at fertility graduation by means of the gamma density. In the latter case, the function which is fitted to the empirical fertility curve has the following form:

$$g(x; R, b, c, d) = \frac{R}{\Gamma(b)\, c^b}(x-d)^{b-1} e^{-(x-d)/c} \quad \text{for } x > d. \tag{4.1}$$

The parameter R still represents the level of fertility, and childbearing starts roughly at age d. If we denote the mean and modal ages of childbearing by μ and m, respectively, and the corresponding variance by σ^2, then

$c = \mu - m,$ and $b = (\mu - d)/c = \sigma^2/c^2.$

Thus, all four parameter R, b, c and d are reasonably interpretable in terms of descriptive characteristics of the fertility curve. It is easy to construct moment estimators. It is also easy to reparametrize, e.g., by substituting, say, R, μ, m, and σ^2 for the four parameters used in (4.1).

4 B. We have carried out some numerical experiments with the gamma density similar to the ones which we described in Subsection 3 D for the Hadwiger function. Thus, we have constructed sets of "observed" fertility rates for a couple of cases, we have calculated corresponding moment "estimates", and we have used these as starting values for an iterative procedure which produces least squares "estimates". The results are roughly as encouraging as were the ones reported in Subsection 3 D. For the cases which we have investigated, a parametrization by R, μ, m, and σ^2 turns out to give much more rapid conver-

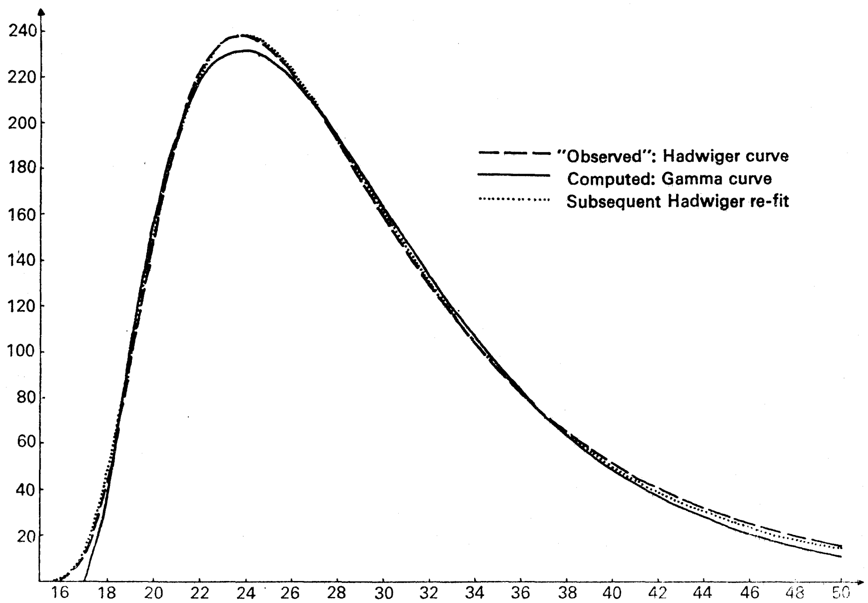

Fig. 5. Gamma graduation of Hadwiger curve and subsequent Hadwiger graduation of computed gamma curve for 13 Norwegian municipalities, 1968–71 (case 1 of Table 5).

gence than, say, the parametrization in (4.1). This is probably due to the greater stability of R, μ, m, and σ^2.

4C. In order to find out how well the gamma density function can represent a Hadwiger curve, we have carried out a gamma least squares "graduation" of the "observed" fertility curve corresponding to the Hadwiger functions with the parameter values of cases 1 and 2 of Table 3. The fitted parameter values of the gamma densities have been listed in Table 5, along with the normed sums of squares of deviations. The curves for case 2 have been plotted in Fig. 4. The fit is very good in this case. The curves for case 1 have been plotted as the stipled curve and the unbroken curve in Fig. 5. The fit is good, but not overwhelmingly so. Out of curiosity, we have subsequently fitted a second Hadwiger curve to the gamma curve of case 1. The resulting parameter values have been listed in parentheses as the "subsequent Hadwiger re-fit" in line 3 of case 1 of Table 5, and the corresponding curve is the dotted one in Fig. 5. This fit gets slightly better. It is interesting to see that the re-fit leaves us roughly where we started.

4D. We have gone on to fit a Hadwiger density by least squares to the "observed" fertility curves corresponding to the gamma density function with the parameter values of cases 2 and 3 of Table 4. The fitted parameter values have been listed in Table 6. The diagrams of these curves look quite similar to Fig. 4 without its stipled curve and are not displayed here. The fit is very good in both cases.

Table 4. *Three sets of values of parameters of the gamma density function, and corresponding probability limits of the moment estimators*

Case no.		R	μ	m	σ^2	b	c	d
1	Parameter value[a]	3.618	29.050	23.773	61.159	2.196	5.277	17.460
	Probability limit	3.553	28.517	25.685	46.924	5.851	2.832	11.948
2	Parameter value[b]	2.939	27.390	23.630	49.056	3.471	3.759	14.341
	Probability limit	2.918	27.188	24.588	43.588	6.448	2.600	10.422
3	Parameter value[c]	1.927	25.120	21.956	32.026	3.198	3.165	15.000
	Probability limit	1.925	25.080	22.314	30.882	4.038	2.766	13.914

[a] Thirteen Norwegian municipalities. (Corresponds to case 1 of Table 3.)
[b] Norway, 1966. (Corresponds to case 2 of Table 3.)
[c] Hungary, 1961. (Gilje, 1969, p. 134.)

4E. To the extent that the results reported in Subsection 4C and 4D carry over to other numerical examples, they raise an important question of principle. If these pairs of curves had appeared as the results of parallel graduations of real data, once by means of the Hadwiger function and once by the gamma density, we know of no objective way of selecting one out of the pair as more appropriate a graduation than the other. In other words, it looks as if we are

Table 5. *Parameter values of the gamma density fitted to the Hadwiger function of cases 1 and 2 in Table 3*

		R	Mean	Mode	Variance	Normed sum of squares of deviations[d]
Case 1	Hadwiger parameter[a]	3.716	29.700	23.669	81.967	
	Gamma parameter	3.618	29.050	23.773	61.159	$4.2 \cdot 10^{-5}$
	(Subsequent Hadwiger re-fit)[b]	(3.695)	(29.460)	(23.792)	(76.001)	$(3.0 \cdot 10^{-5})$
Case 2	Hadwiger parameter[c]	2.957	27.502	23.628	53.453	
	Gamma parameter	2.928	27.322	23.687	48.254	$1.0 \cdot 10^{-5}$

[a] Thirteen Norwegian municipalities, 1968–71 (case 1 of Table 3).
[b] For an explanation, see Subsection 4C.
[c] Norway, 1966 (case 2 of Table 3).
[d] For an explanation of the norming procedure, see Subsection 2F.

Table 6. *Parameter values of the Hadwiger density fitted to the gamma densities of cases 2 and 3 in Table 4*

Case no.[a]		R	Mean	Mode	Variance	Normed sum of squares of deviations[d]
2	Gamma parameter[b]	2.939	27.390	23.630	49.056	
	Hadwiger parameter	2.967	27.546	23.603	54.069	$1.2 \cdot 10^{-5}$
3	Gamma parameter[c]	1.927	25.120	21.956	32.026	
	Hadwiger parameter	1.947	25.258	21.937	35.654	$1.9 \cdot 10^{-5}$

[a] For case 1, see Table 5 and Subsection 4C.
[b] Norway, 1966.
[c] Hungary, 1961.
[d] For an explanation of the norming procedure, see Subsection 2F.

faced with a second practical identification problem connected with the Hadwiger function, over and above the one reported in Subsection 2E above. The second problem is that of distinguishing the values of the Hadwiger function from corresponding values of a gamma density in a manner sufficiently clear to be of practical use in applications to real data.

This finding fits well in with the fact that previous research has failed to establish either of these functions as systematically superior to the other for purposes of fertility graduation.

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Addendum

Ole Barndorff-Nielsen and Søren Johansen have brought to our attention the fact that the Hadwiger density discussed in Section 2 [above] has appeared in the statistical literature before. Johnson & Kotz (1970) devote a chapter called "Inverse Gaussian (Wald) distributions" to it, and later papers by Wani & Kabe (1973) and Basu & Wasan (1974) contain further material.

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Nokre praktiske røynsler med analytisk glatting

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0. Samandrag

I samband med ein studie av regional fertilitet i Norge ca. 1970 har forfattarane freista å undersøke om Hadwiger-funksjonen kan seiast å vere systematisk betre eller dårlegare som glattingsfunksjon enn GG-funksjonen, som er basert på gammatettheta. Vi har ikkje funne noko eintydig svar. GG-funksjonen kjem best ut i svært mange høve. Hadwiger-funksjonen synest likevel å vere meir fleksibel, og mykje tyder på at han er den overlegne av dei to når fertilitetskurva er venstreskeiv og har tung høgrehale.

Vi har og vilja sjå i kva mon ulike val av øvre og nedre grense for dei aldersgruppene som er med under glattinga, gir ulike glatta fertilitetskurver. Det viser seg at dette valet kan ha stor verknad på resultatet.

1. Innleiing

1.1. Prosjektet

Forfattarane av dette notatet har gjort ein studie av regional fertilitet i Norge i åra rundt 1970. Dei 449 kommunane i landet 1/1 1971 er grupperte i 77 fertilitetsområde på grunnlag av næringsstruktur, geografisk plassering og om dei har systematisk inn- eller utflyttning. Kvart område vil typisk ha 30 000—50 000 innbyggjarar. For kvart område har vi rekna ut fødselsrater for kvar einskild alder frå 15 år til 50 år ved å aggregere data for åra 1968 til 1971. Når ein teiknar eit diagram av eit sett av slike rater med alderen langs abscisseaksen, får ein fram ei "fertilitetskurve". Kvar slik kurve har vi glatta analytisk ved å tilpasse ein Hadwiger-funksjon til henne. For nokre av dei har vi og laga ei alternativ glatting med ein GG-funksjon, som er ein glattingsfunksjon basert på gammatettheta.

Vi har freista å undersøke om Hadwiger-funksjonen kan seiast å vere systematisk betre eller dårlegare enn GG-funksjonen. Vi har og vilja sjå i kva mon ulike val av øvre og nedre grense for dei aldersgruppene som vi tek med i glattinga, gir ulike glatta fertilitetskurver. I dette notatet skal vi rapportere om praktiske røynsler vi har gjort i dette arbeidet. Dei resultata vi legg fram her, er eit representativt utval av dei vi har funne under denne delen av prosjektet. Nokre fleire detaljer finst i Berge og Hoem (1974).

Tidlegere rapportar frå andre delar av prosjektet har vi gitt i Berge (1973, 1974) og Hoem og Berge (1974 a, b).

1.2. Fertilitetskurvene

Formelen vi har nytta til å rekne ut fødselsraten for x-årige kvinner i fertilitetsområde nr. i, er

$$ {}_i\hat{\lambda}_x = \frac{\sum_{n=1968}^{1971} {}^n_iF_x}{\sum_{n=1968}^{1971} \tfrac{1}{2}\left\{{}^{n-1}_{\;\;i}L_{x-1} + {}^n_iL_x\right\}} = \frac{{}_iF_x}{{}_iM_x}, $$

der $x = 15, 16, \ldots, 50$ og $i = 1, 2, \ldots, 77$. Her er ${}^n_iF_x$ talet på levandefødde barn i

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året n i kommunegruppe nr. i av mødrer som ved utgangen av året n vil vere x år gamle, og ${}^n_iL_x$ er talet på kvinner i alt i befolkninga i kommunegruppe nr. i som ved utgangen av året n er x år. I diagramma er ratene oppgjevne i promille.

For å teste eigenskapane til glattingsfunksjonane valde vi ut fem av områda slik at vi skulle få representert dei viktigaste kurvetypene vi har støytt på i materialet. Desse kommunegruppene har nummera 10, 32, 38, 43 og 77. Dei er samansett slik ein ser av tab. 1.

Oslo er det største av fertilitetsområda våre, rekna etter folketalet. Område nr. 77 (kommunar i Finnmark) er det minste.

Tab. 1. Kommunane i fertilitetsområda 10, 32, 38, 43 og 77

Fertili-tetsområde nr.	Kommunar		Samla folketal 31/12 1970
10	0301	Oslo	482 000
32	0806	Skien	45 000
38	1004	Flekkefjord	
	1101	Eigersund	
	1102	Sandnes	50 000
43	1144	Kvitsøy	
	1145	Bokn	
	1151	Utsira	
	1244	Austevoll	
	1245	Sund	
	1259	Øygarden	
	1264	Austrheim	
	1265	Fedje	
	1412	Solund	
	1428	Askvold	
	1441	Selje	
	1514	Sande	
	1532	Giske	31 000
77	2011	Kautokeino	
	2020	Porsanger	
	2021	Karasjok	
	2025	Tana	
	2027	Nesseby	13 000

1.3. Glattingsfunksjonane

Med den tradisjonelle parametriseringa ser Hadwiger-funksjonen slik ut:

$$h_o(x;R,H,T,U)=\frac{RH}{T\sqrt{\pi}}\left(\frac{T}{x-U+T}\right)^{-3/2}$$

$$\exp\left\{-H^2\left(\frac{T}{x-U+T}+\frac{x-U+T}{T}-2\right)\right\}$$

for $x>U-T$.

Her tolkar vi parameteren R (arealet under kurva) som det samla fertilitetstalet. U tolkar vi som gjennomsnittleg fødealder. Parametrane H og T kan vi derimot ikkje gi ei substansiell demografisk tolking.

I eit tidlegare notat (Hoem og Berge, 1974 a) har vi vist at denne parametriseringa gir oss eit praktisk identifikasjonsproblem ved at store variasjonar i parametrane ikkje treng føre til meir enn små endringar i dei tilsvarande kurvene. Det ser ut til at vi kan løyse desse problema ved å erstatte parametrane H og T med to nye parametrar, S^2 og M, der S^2 er variansen i sannsynlighetsfordelinga med tettheta $h_o(.;R,H,T,U)/R$, medan M er modalverdien til funksjonen.

Den tilsvarande "klassiske" parametriseringa av GG-funksjonen er

$$g_o(x;R,b,c,d)=\frac{R}{\Gamma(b)c^b}(x-d)^{b-1}$$

$$\exp\{-(x-d)/c\}\quad \text{for } x>d,$$

$$g_o=0 \qquad\qquad \text{for } x\leqslant d.$$

Her og løner det seg å reparametrisere, slik at ein brukar parametrane R, m, μ og σ^2 i glattinga, der m, μ og σ^2 er modalverdien, forventningsverdien og variansen i gammatettheta.

Vi har difor brukt glattingsfunksjonane $h(x;\Theta)$ og $g(x;\Theta)$, der h er Hadwiger-funksjonen parametrisert med $\Theta=(R,M,U,S^2)$ medan g er GG-funksjonen med $\Theta=(R, m,\mu,\sigma^2)$. Vi har samanlikna tilpassingsevna til desse to funksjonane.

1.4. Glattingsmetode

La $\{\hat{\lambda}_x: x = \alpha, \alpha+1, \ldots, \beta\}$ vere ei rekkje fødselsrater for aldrane α, $\alpha+1, \ldots, \beta$, og la $f(x;\Theta)$ vere ein valt glattingsfunksjon. Å glatte fertilitetskurva til desse ratene analytisk vil då seie at ein vel ein verdi $\hat{\Theta}$ av parameteren Θ slik at kurva av glatta verdiar $\{f(x;\hat{\Theta}) : x = \alpha, \alpha+1, \ldots, \beta\}$ i ei eller anna meining høver godt med fertilitetskurva $\{\hat{\lambda}_x: x = \alpha, \alpha+1, \ldots, \beta\}$. Sidan $\hat{\Theta}$ må veljast på basis av observasjonane, reknar vi han som ein statistisk estimator.

Det finst mange måter å velje $\hat{\Theta}$ på. Hoem (1972) har likevel før vist at det kan løne seg å estimere Θ ved hjelp av ein modifisert kji-kvadratminimeringsmetode. Det vil seie at ein som $\hat{\Theta}$ vel den verdien av Θ som gjer

$$\sum_{x=\alpha}^{\beta} \frac{\{\hat{\lambda}_x - f(x;\Theta)\}^2}{\hat{\lambda}_x/M_x}$$

minst mogeleg. Ein får da minimert den asymptotiske variansen av estimatoren $\hat{\Theta}$ og av $f(x;\hat{\Theta})$ for alle x (innanfor ei stor klasse av mogelege estimatorar). Vi har óg vist (Hoem og Berge, 1974 b) at denne metoden i praksis gir merkbart lågare asymptotisk varians enn om ein til dømes nyttar minste kvadraters metode og minimerer

$$\sum_{x=\alpha}^{\beta} \{\hat{\lambda}_x - f(x;\Theta)\}^2.$$

For å kunne jamføre, har vi i dette arbeidet nytta begge desse metodane.

1.5. Val av aldersklasser i glattinga

Når ein nyttar eittårige aldersintervall, reknar ein til vanleg ut fødselsratene for aldrane frå 15 til 44 år, eller frå 15 til 49 år eller 50 år, etter som det høver. La oss kalle lågaste og høgste alder for a og b.

Når ein skal glatta fertilitetskurva som svarer til ratene $\{\hat{\lambda}_x : x = a, a+1, \ldots, b\}$, treng ein naturlegvis ikkje ta med alle desse aldersklassane i reknearbeidet. Ein kan til dømes nøye seg med å ta med ratene $\{\hat{\lambda}_x: x = \alpha, \alpha+1, \ldots, \beta\}$ under glattinga, der $a \leqslant \alpha < \beta \leqslant b$. Når ein så har funne den verdien $\hat{\Theta}$ av parametrane ein vil bruke, kan ein sjølsagt likevel rekna ut dei glatta fødselsratene $f(x, \hat{\Theta})$ for alle x frå $x = a$ til $x = b$.

I nokre høve er ein heilt enkelt nøydd til å velje slike α og β som ikkje fell saman med a og b. For fertilitetsområda våre har vi brukt $a = 15$ og $b = 50$, men vi får ofte $\hat{\lambda}_x = 0$ for høge aldrar x, og nokre gonger er og $\hat{\lambda}_{15} = 0$. Skal ein bruke kjikvadrat-minimering, nyttar det ikkje å ta med aldrar x der $\hat{\lambda}_x = 0$, for da har ikkje formelen vi gav i avsnitt 1.3 noka meining. Med kjikvadratminimeringsmetoden likar ein i det heile ikkje få fødslar i ei aldersklasse. Eit vanleg kriterium er å krevje at $F_x > 5$ for alle dei x ein tar med. Som ein vil sjå i kapittel 2 nedanfor, har vi nytta dette kriteriet. Vi har der studert i kva mon val av α og β har merkbar verknad på glattinga.

1.6. Kvalitetskriteria

Ved rota av dei problema vi tar opp her, ligg dette spørsmålet:

Kva skal ein krevje av ei glatting for å seie at ho passar godt til data?

Når glattingsfunksjonen er gitt, er svaret altså gitt om vi går ut fra at minimal asymptotisk varians er eit dominerande krav. Estimeringa må da baserast på den modifiserte kjikvadratminimeringsmetoden (eller ein annan metode med like gode eigenskapar). Men når vi skal velje mellom ulike forslag til glattingsfunksjonar, treng vi kriterium ut over dette.

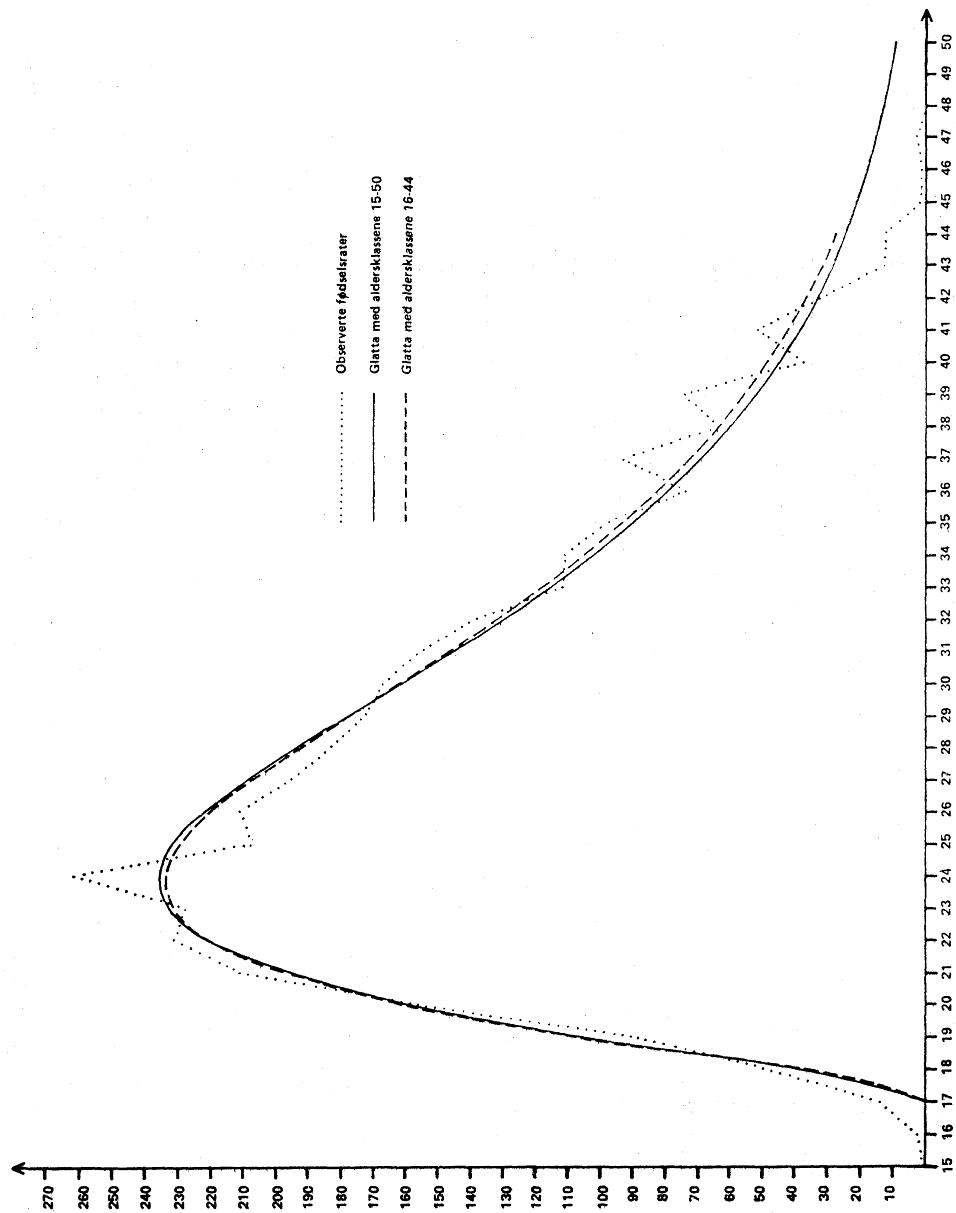

Fig. 1. Fertilitetskurva for område nr. 43 glatta med GG-funksjonen og minste kvadraters metode og med ulike aldersklasser

Eitt rimeleg slikt kriterium er at kjikvadratverdien eller kvadratavviksummen for tilpassinga skal vere så liten som mogeleg. Det kan óg vere at vi ønskjer særleg god tilpassing i dei sentrale (mest fruktbare) aldrane, til dømes frå 20 til 30 år. Vi må da sjå på kjikvadratverdi eller kvadratavviksum for tilpassinga mellom desse aldrane for seg.

Vi skal sjå kva resultat desse ideane gir i det følgjande.

1.7. Hovudkonklusjonar

(i) Det viser seg at tilpassing med GG-funksjonen gir lågare kjikvadratverdi enn om ein nyttar Hadwiger-funksjonen i tre firedelar av dei numeriske døma våre. På mange måtar synest Hadwiger-funksjonen likevel å passe betre både i dei sentrale aldersklassene og i dei aller lågaste aldersklassene vi tek med i utrekningane. Vi har og inntrykk av at Hadwiger-funksjonen stort sett høver best av dei to når fertilitetskurva er sterkt venstreskeiv og har ein "tung" hale til høgre. Dette går vi nærare inn på i kapittel 3 nedanfor.

(ii) Når glattingsfunksjonen er valgt, kan ein få store variasjonar i estimatet av Θ og i dei glatta kurvene etter kva aldersklasser ein tar med i glattinga. Dette synest å gjelde særleg når vi nyttar kjikvadratmetoden i estimeringa. Det er mindre merkbart med minste kvadraters metode. Kjikvadratmetoden vektar dei små observasjonane $\hat{\lambda}_x$ mest. (Sjå formelen i avsnitt 1.3 ovanfor.) Sidan fødselsratene $\hat{\lambda}_x$ stort sett er nær null når x kjem over 45 år, vil ein presse den høgre halen av funksjonen lenger ned mot aksen di fleire aldersklasser over 44 år ein tar med. (Noko tilsvarande gjeld visst ikkje for dei lågaste aldersklassene.) Vi drøfter dette i kapittel 2 nedanfor.

2. Val av aldersklasser i glattinga

2.1. Aldersklassene ved kjikvadratminimering

I avsnitt 1.4 ovanfor nemnde vi at ein treng å legge omtanke i kva for aldersklasser ein tar med i utrekningane. La oss "estimere" den lågaste aldersklassen, α, og den høgste, β, vi vil ta med ved kjikvadratminimering ved å sette

$$\hat{\alpha} = \min \left\{x: \hat{\lambda}_x M_x > 5 \ \& \ \hat{\lambda}_y M_y \leqslant 5 \text{ for alle } y < x\right\}$$

og

$$\hat{\beta} = \max \left\{x: \hat{\lambda}_x M_x > 5 \ \& \ \hat{\lambda}_y M_y \leqslant 5 \text{ for alle } y > x\right\}.$$

(Vi nyttar i praksis $\hat{\lambda}_x M_x$ i staden for F_x for å sleppe med å lese inn $\hat{\lambda}_x$-ane og M_x-ane i reknemaskinprogrammet.) Dette har vi gjort for alle dei 77 fertilitetsområda. I tab. 2 er frekvensfordelinga for $\hat{\alpha}$ og $\hat{\beta}$ gitt.

Tab. 2. Frekvensfordeling for "estimat" av yttergrensene for reproduktiv periode

$\hat{\alpha}$	ant. område	$\hat{\beta}$	ant. område
15	2	41	1
16	35	42	13
17	40	43	15
		44	33
		45	12
		46	3

Når ein skal nytte faste grenser for dei aldersklassene ein tar med i glattinga, viser desse tala at aldersintervallet 16—44 år vil vere eit naturleg val av α og β i Norge i dag. Denne konstateringa dekker imidlertid over at $(\hat{\alpha}, \hat{\beta})$ kan variere frå 15—46 i eit område til 17—41 i eit anna.

Vi må også vere klar over at $\hat{\alpha}$ og $\hat{\beta}$ ikkje berre er biologisk bestemt, men og sosialt. Særleg vil $\hat{\beta}$ variere områda imellom innan vår samfunnstype.

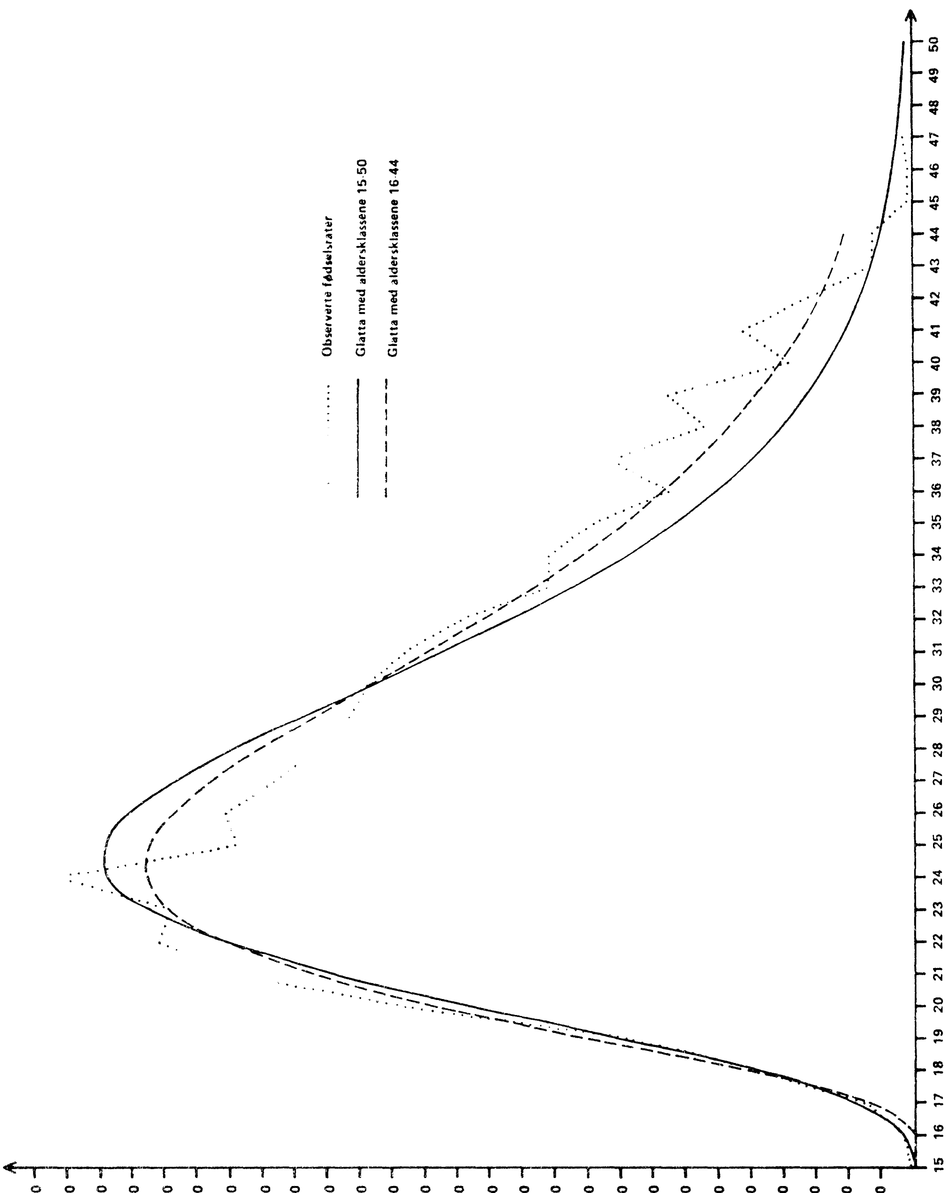

Fig. 2. Fertilitetskurva for område nr. 43 glatta med GG-funksjonen og kjikvadratminimering og med ulike aldersklasser

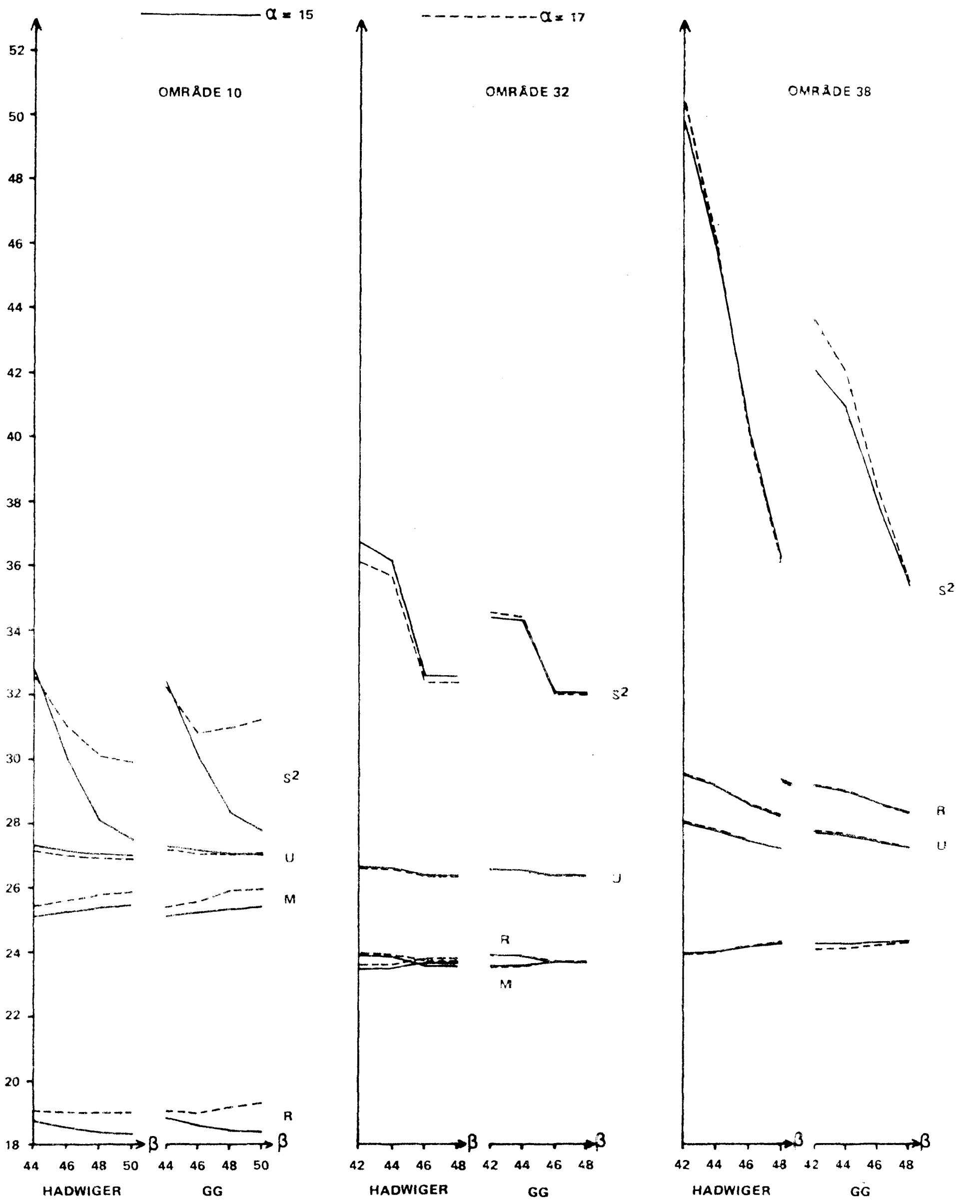

Fig. 3. Estimat av parametrane i Hadwiger- og GG-funksjonen for fertilitetsområda 10, 32 og 38 ved bruk av ulike aldersklasser

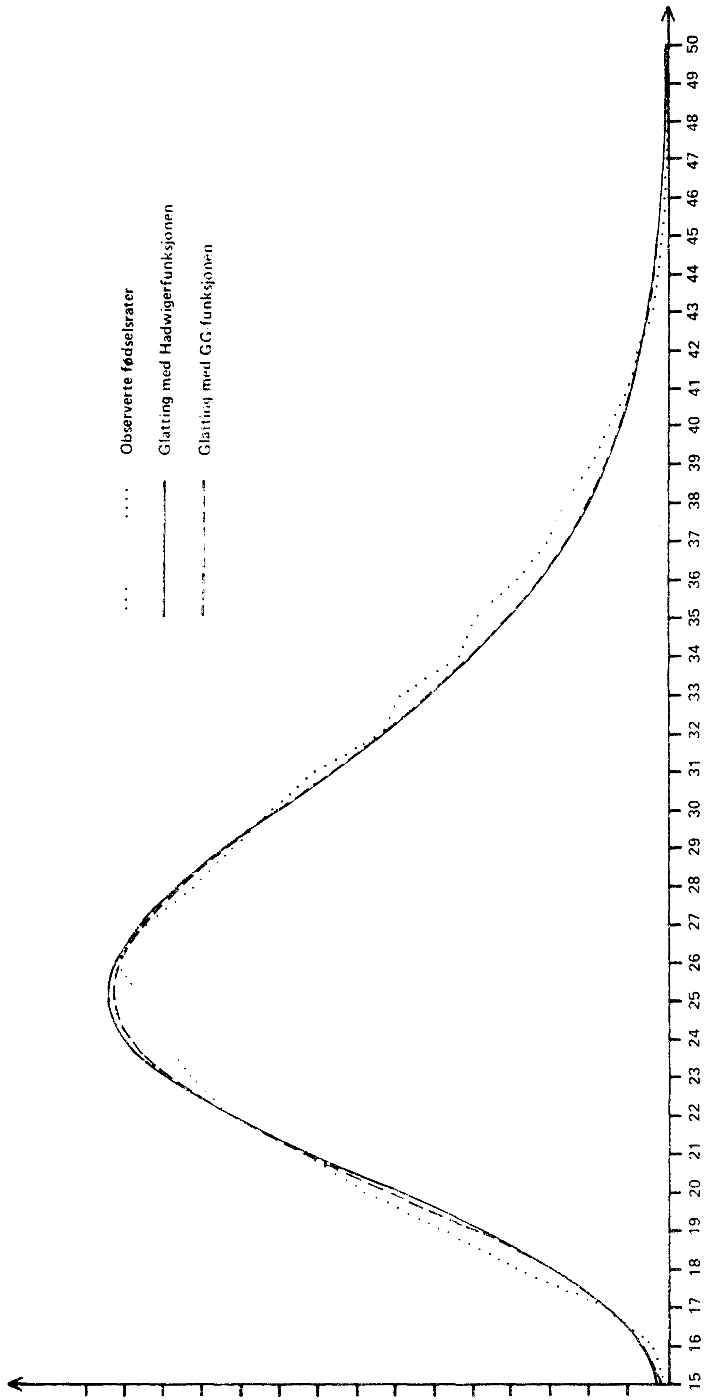

Fig. 4. Fertilitetskurva for område nr. 10 glatta ved kjikvadratminimering med aldersklassene fra $\hat{\alpha}=15$ til $\hat{\beta}=46$

2.2. Kurve- og parameterestimat

For å studere kva effekt ulike val av α og β har på parameterestimatet $\hat{\Theta}$ og på dei glatta kurvene, tok vi føre oss dei fem områda som er nemnde i tab. 1. Dei "estimerte" verdiane for α og β vart slik ein ser av tab. 3.

Tab. 3. $\hat{\alpha}$ og $\hat{\beta}$ for fertilitetsområda 10, 32, 38, 43 og 77

Område nr.	$\hat{\alpha}$	$\hat{\beta}$
10	15	46
32	16	44
38	17	44
43	17	44
77	17	42

For kvart av desse områda estimerte vi θ med kjikvadratminimeringsmetoden med α fast lik 15, 16 og 17 mens β varierte frå $\hat{\beta}-2$ i skritt på 2 til $\hat{\beta}+4$. Nokre av resultata er framstilte grafisk i figurane 1 til 3. I figurane 1 og 2 har vi tatt for oss fertilitetsområde 43 serskilt. Vi har plotta både den observerte fertilitetskurva og dei glatta kurvene ein får når ein tar med aldersklassene 15—50 og alternativt klassene 16—44 og nyttar GG-funksjonen i utjamninga. Diagramma for Hadwiger-funksjonen er heilt analoge. Vi har og brukt både kjikvadratminimeringsmetoden og alternativt minste kvadraters metode i glattingane. Det er tydeleg at valet av α og β har stor verknad på utsjånaden av kurvene når ein bruker kjikvadratminimeringsmetoden. Om ein nyttar minste kvadraters metode, ser ikkje valet av α og β ut til å spele så stor rolle. Dette gjeld både når ein nyttar Hadwiger-funksjonen og når ein legg GG-funksjonen til grunn. Vi får tilsvarande resultat med kurvene for dei fire andre fertilitetsområda.

I fig. 3 har vi plotta nokre parameterestimat for $\alpha = 15$ og 17 og for $\beta = \hat{\beta}-2, \hat{\beta}, \hat{\beta}+2$ og $\hat{\beta}+4$ for begge glattingsfunksjonane.

Om vi held β fast og ser på effekten av å variere α, finn vi at han for det meste er forsvinnande liten. Den stipla og den heiltrekte kurva fell omtrent saman i kvart kurvepar. For område nr. 10, der $\alpha > \hat{\alpha}$ når $\alpha = 17$, får vi likevel større utslag av endringane i α enn av endringane i β.

Ser vi på effekten av å endre β for fast α, finn vi at denne er stor for variansparameteren, men heller liten for dei andre parametrane, særleg for modal fødealder.

Parameterestimata oppfører seg om lag likt for dei to glattingsfunksjonane. Men variansparameteren i Hadwiger-funksjonen reagerer kraftigare på endringane i β enn den tilsvarande parameteren gjer i GG-funksjonen. Likevel vil variansparameteren i Hadwiger-funksjonen stort sett vere større i numerisk verdi enn variansparameteren i GG-funksjonen.

At Hadwiger-funksjonen reagerer kraftigare når ein tar med nye fødselsrater som ligg nær null, kan vi tolke som eit utslag av større fleksibilitet hos han. På den andre sida vil den større variansparameteren i Hadwiger-funksjonen føre til at vi som regel får nokså høge verdiar av $h(x;\hat{\Theta})$ for $x > \beta$, noko som i mange høve kan vere ein uheldig eigenskap ved funksjonen dersom ein treng glatta fødselsrater for desse aldrane. Når vi nyttar Hadwiger-funksjonen og treng $h(x;\hat{\Theta})$ for $x \geqslant \hat{\beta}$, bør vi sikre oss at β i alle fall ikkje er mindre enn $\hat{\beta}$ sidan fødselsratene til vanleg vil bli overestimert dersom vi har $\beta < \hat{\beta}$. For $\beta > \hat{\beta}$ finn vi oftast små skilnader mellom variansparametrane i dei to funksjonane samanlikna med den skilnaden vi finn for $\beta < \hat{\beta}$.

Generelt kan vi konkludere med at dei numeriske resultata våre tyder på at vi bør sikre oss at $\alpha \leqslant \hat{\alpha}$ og $\beta \geqslant \hat{\beta}$ anten vi nyttar Hadwiger- eller GG-funksjonen.

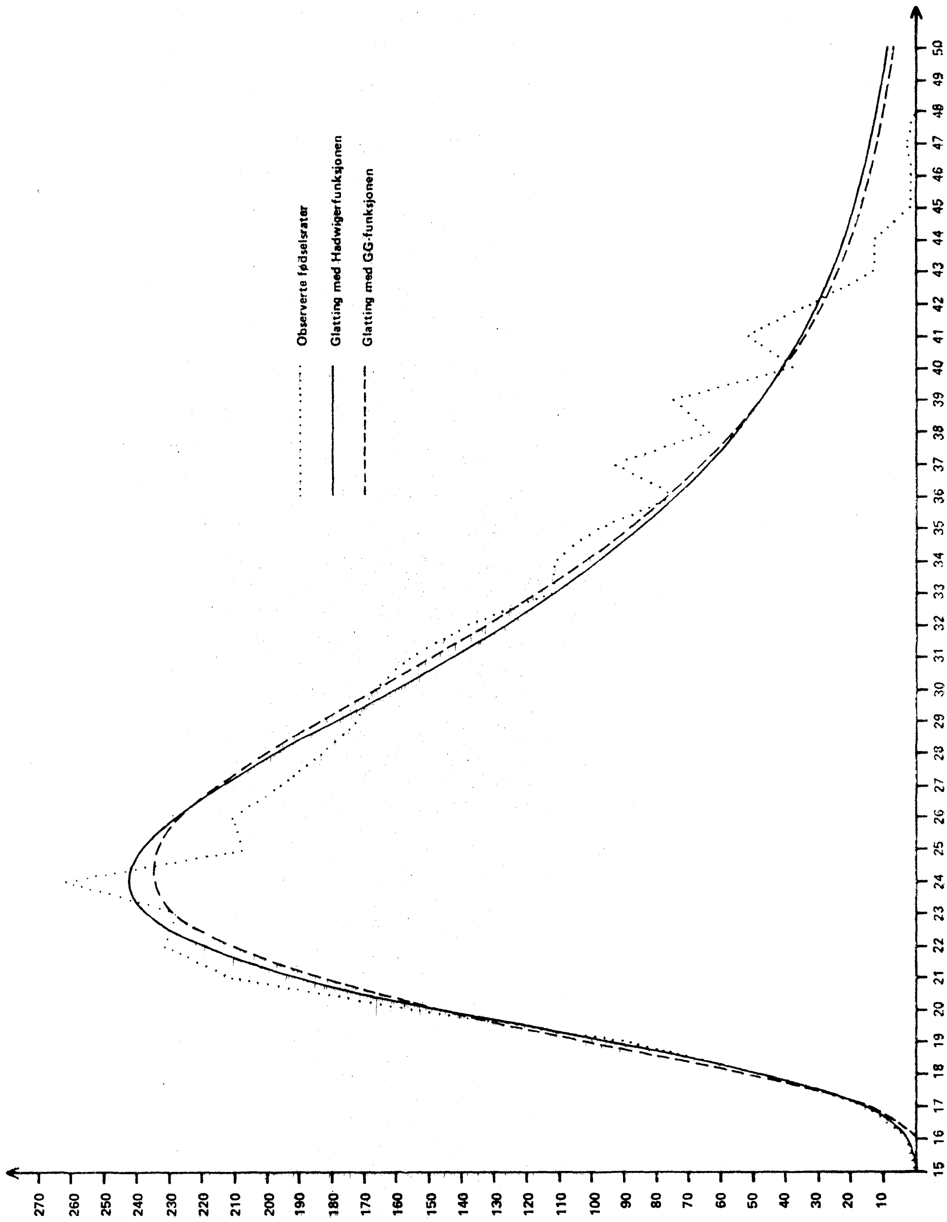

Fig. 5. Fertilitetskurva for område nr. 43 glatta ved kjikvadratminimering med aldersklassene fra $\hat{\alpha}=17$ til $\hat{\beta}=44$

3. Samanlikningar mellom Hadwiger-tilpassing og GG-tilpassing

3.1. Innleiing

Dei utrekningane vi omtala først i avsnitt 2.2, gir oss i alt 60 samanlikningar mellom glattingar med Hadwiger-funksjonen og glattingar med GG-funksjonen. For kvart av dei fem områda får vi tolv glattingar for kvar funksjon. Vi skal no gi eit oversyn over desse samanlikningane.

3.2. Fertilitetskurvene

I fig. 4–6 har vi gitt eit utval av dei 60 par av glattingar vi har utført. Det er ikkje lett å plukke ut ein av dei to funksjonane på augnemål og seie at han er systematisk betre enn den andre. Dei to funksjonane ser for det meste ut til å vere om lag like gode. Fig. 5 og 6 kan likevel tyde på at Hadwiger-funksjonen er best av dei to når fertilitetskurva er sterkt venstreskeiv.

Tab. 4. Samanlikning mellom Hadwiger-tilpassing og GG-tilpassing etter tre tilpassingskriterium

	Kvadratavviksum		Kjikvadratverdi
	Alder 20–30 år	Alder 15–50 år	Alder $a-\beta$
Prosent av samanlikningane der GG-tilpassinga er best	50	78	75
Talet på samanlikningar (N)	60	60	60

Tab. 5. Verdiar av observerte og glatta fødselsrater pr. 1000 for aldrane 15, 16 og 17 for fertilitetsområda 10, 32, 38, 43 og 77 (sjå også figurane 1, 2, 4, 5 og 6)

Fertilitetsområde nr.	Aldersklassar med i glattinga	Glattingsfunksjon[1]	Alder 15	Alder 16	Alder 17
10	Observert fødselsrate		0,67	3,75	15,39
	15–46	HADW.	2,10	6,27	14,63
		GG.	1,60	5,89	14,88
32	Observert fødselsrate		2,85	4,36	21,14
	16–44	HADW.	0,64	6,16	23,43
		GG.	0	4,18	24,75
38	Observert fødselsrate		0,61	3,11	14,52
	17–44	HADW.	0,13	3,09	16,94
		GG.	0	0,05	14,61
43	Observert fødselsrate		0,82	2,57	14,43
	15–50	HADW.	0,55	4,49	17,67
		GG.	0,00	2,67	17,51
		HADW. (LSQ)	0,00	1,06	13,85
		GG. (LSQ)	0	0	0
	16–44	HADW.	0,04	2,14	15,78
		GG.	0	0	12,65
		HADW. (LSQ)	0,00	0,19	9,12
		GG. (LSQ)	0	0	0
	17–44	HADW.	0,02	1,87	15,10
		GG.	0	0	12,64
77	Observert fødselsrate		0	3,75	21,11
	17–42	HADW.	0,04	3,44	23,23
		GG.	0	0	20,78

[1] Der det ikkje er nytta kjikvadratminimering men minste kvadrater under parameterestimeringa, er dette særskilt avmerka med LSQ.

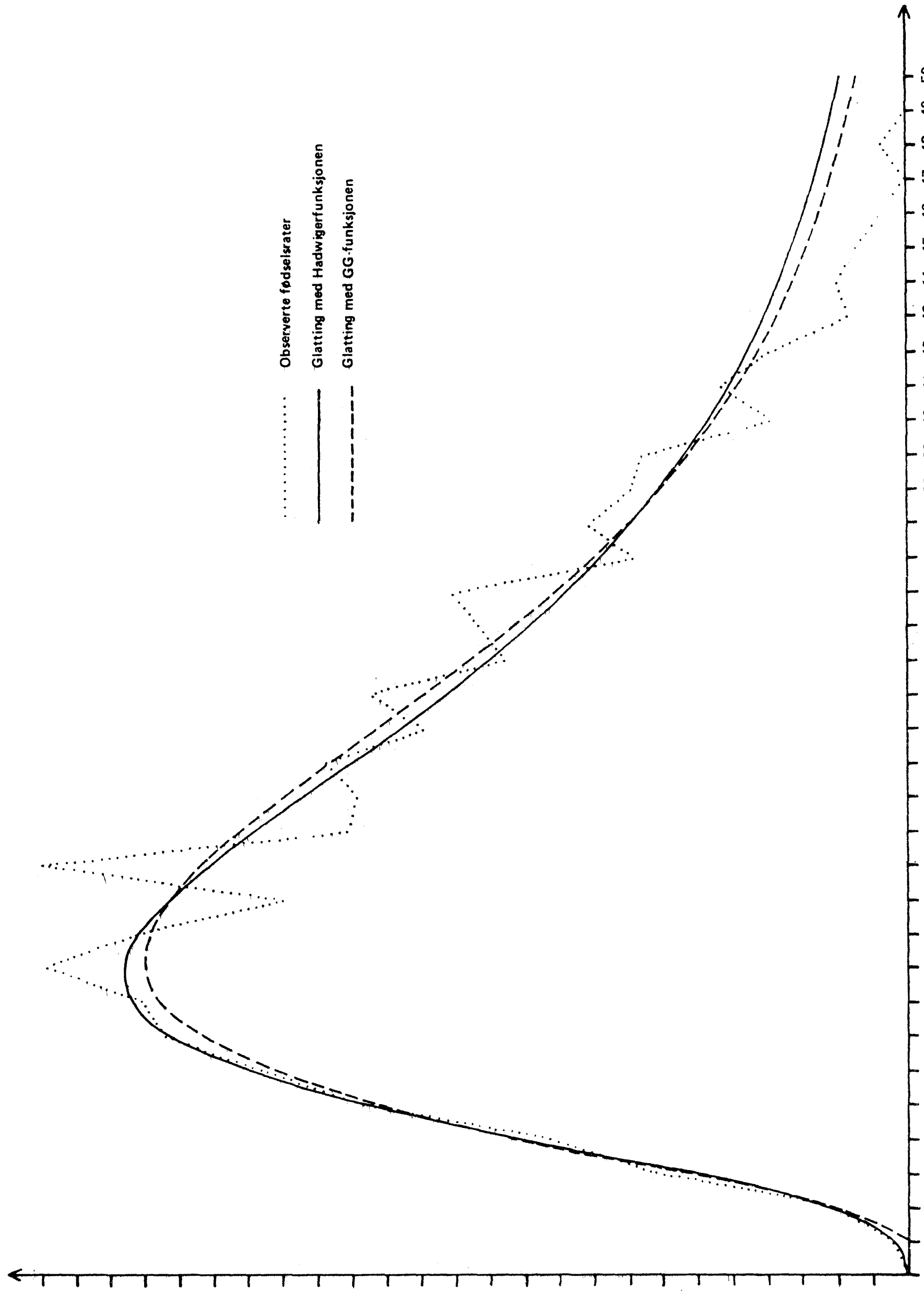

Fig. 6. Fertilitetskurver for område nr. 77 glatta ved kjikvadratminimering med aldersklassene fra $\hat{\alpha}=17$ til $\hat{\beta}=42$

Tab. 6. Kvadratavviksum og kjikvadratverdi for Hadwiger-tilpassing i prosent av same mål ved tilsvarande GG-tilpassing for utvalde fertilitetsområde [2, 3]

Område nr.		10			32			38			43			77		
Aldersintervall for estimering av Θ		Kvadrat-avviksum		Kji-kva-drat-verdi	Kvadrat-avviksum		Kji-kva-drat-verdi	Kvadrat-avviksum		Kji-kva-drat-verdi	Kvadrat-avviksum		Kji-kva-drat-verdi	Kvadrat-avviksum		Kji-kva-drat-verdi
		Alder 20–30	Alder 15–50	Alder $\alpha-\beta$	Alder 20–30	Alder 15–50	Alder $\alpha-\beta$	Alder 20–30	Alder 15–50	Alder $\alpha-\beta$	Alder 20–30	Alder 15–50	Alder $\alpha-\beta$	Alder 20–30	Alder 15–50	Alder $\alpha-\beta$
α	β															
15	40													(1)	(1)	(1)
15	42				119	126	123	79	93	84	67	100	74	91	111	89
15	44	134	136	138	120	128	124	80	93	95	91	104	100	87	106	105
15	46	137	130	127	125	132	127	86	101	111	113	115	113	93	109	115
15	48	134	123	119	125	132	127	94	110	117	114	116	113			
15	50	133	121	117												
17	40													(1)	(1)	(1)
17	42				116	119	126	84	97	97	60	100	77	90	111	100
17	44	115	117	112	117	120	126	85	98	107	87	103	108	95	107	111
17	46	116	114	108	120	122	123	94	106	116	126	116	111	102	109	115
17	48	80	99	140	120	122	123	103	112	117	127	116	111			
17	50	83	105	150												
GG-tilpassing best[4]		75 %	88 %	100 %	100 %	100 %	100 %	13 %	50 %	63 %	50 %	75 %	63 %	13 %	75 %	50 %

[1] I tre høve klarte vi ikkje å estimere parametrane til GG-funksjonen. Det skjedde i område 77 for $\beta = 40$ og $\alpha = 15$, 16 og 17.
[2] Tomme celler svarer til høve der vi ikkje har rekna ut noko.
[3] Resultata for $\alpha = 16$ er ikkje tatt med her.
[4] Talet på samanlikningar (N) er 8 heile vegen.

3.3. Numeriske samanlikningskriterium

For å få meir objektive samanlikningskriterium enn reint augnemål, har vi rekna ut verdet av tre mål til å samanlikne tilpassingane. For kvar glatting har vi rekna ut kvadratavviksummen for aldrane 20—30 år og for aldrane 15—50 år, og kjikvadratverdien for aldrane mellom α og β. For kvar Hadwiger-tilpassing har vi så rekna ut kor stort kvart av desse tre måla er i prosent av same mål ved den tilsvarande GG-tilpassinga. Eit grovt oversyn over utfallet av samanlikningane får ein av tab. 4 på side 304.

Når ein ser på kvadratavviksummen for aldrane frå 20 til 30 år, er dei to funksjonane omtrent like gode for dei døma vi har studert. Etter dei to andre kriteria er GG-funksjonen best i dei fleste høve.

3.4. Systematiske mønster

Det er viktig å finne ut om det er noko systematisk mønster i dei situasjonane der Hadwiger-funksjonen likevel er best. For å undersøke det, ser vi på resultata for dei einskilde områda. Nokre av desse resultata finn ein i tab. 6.

Det synest som om GG-funksjonen kjem betre ut av samanlikninga di større β er. Hadwiger-funksjonen er gjennomgående best der kurvene er sterkt venstreskeive og har "tunge" høgrehaler, slik som for fertilitetsområda 38, 43 og 77. I desse områda ser det ut til at Hadwiger-funksjonen særleg er overlegen i dei sentrale aldrane.

3.5. Dei lågaste aldrane

Når ein nyttar GG-funksjonen, vil ein ofte få ei glatta fødselsrate på 0 for dei aller lågaste aldrane av di $g(x;\hat{\Theta}) = 0$ for $x \leqslant \hat{\mu} - \hat{\sigma}^2/(\hat{\mu} - \hat{m})$. (I tab. 5 finn ein verdien for $g(x;\hat{\Theta})$ for $x = 15$, 16 og 17 for alle kurvene i dette notatet.) Figurane 1, 5 og 6 og andre slike figurar tyder på at den tilpassinga ein får med GG-funksjonen blir dårlegare enn med Hadwiger-funksjonen for fødselsratene i dei aldrane der $g(x;\hat{\Theta}) = 0$.

Tab. 7 tyder på at dei to funksjonane er om lag jamgode i asymptotisk varians av dei glatta fødselsratene for aldrane over dei aller yngste, men for små x er Hadwiger-funksjonen best etter dette kriteriet òg. (Utrekning av asymptotisk varians er forklart av Hoem og Berge, 1974 b.)

Tab. 7. Variasjonskoeffisienten for den glatta fødselsraten i Hadwiger-tilpassinga når $\alpha = \hat{\alpha}$ og $\beta = \hat{\beta}$ i prosent av variasjonskoeffisienten i den tilsvarande GG-tilpassinga

Område nr.	10	32	38	43	77
Alder					
16	89	61	5	(1)	(1)
17	98	103	78	74	81
20	99	106	101	104	108
24	103	99	100	101	99
28	101	108	105	105	107
32	104	104	106	108	105
37	101	88	99	99	98
44	103	102	102	101	103

[1] Både fødselsraten og den asymptotiske variansen er estimert til 0 under GG-tilpassinga her.

4. Sluttmerknad

I tidlegare granskingar har ein ikkje kunna legge fram avgjerande argument verken av teoretisk eller praktisk art for å velje ein gong for alle mellom Hadwiger-funksjonen og GG-funksjonen. Det ser ikkje ut til at vi kan gjere det på grunnlag av dei resultata som er lagde fram her heller. Vi har før hatt eit inntrykk av at Hadwiger-funksjonen er noko meir fleksibel enn GG-funksjonen, slik at han gir ei akseptabel tilpassing til eit breiare spektrum av fertilitetskurver. Dette inntrykket sit vi fortsatt med. Elles må ein seie at GG-funksjonen passar best til nokre fertilitetsmønster, medan Hadwiger-funksjonen har føremoner i samband med andre, særleg slike mønster som gir ei venstreskeiv kurve med ein tung høgrehale.

5. Litteraturliste

[1] Berge, Erling (1973): "Samfunnsstruktur og fruktbarhet". Upublisert hovedfagsoppgave. Universitetet i Bergen og Statistisk Sentralbyrå, Oslo.

[2] Berge, Erling (1974): "MINSYS: Eit reknemaskinprogram for analytisk glatting av befolkningsrater". Statistisk Sentralbyrå, ANO IO 74/11.

[3] Berge, Erling og Jan M. Hoem (1974): "Nokre praktiske røynsler med analytisk glatting". Statistisk Sentralbyrå, ANO IO 74/23.

[4] Hoem, Jan M. (1972): "On the statistical theory of analytic graduation". Proc. Sixth Berkeley Symp. Math. Statist. Prob., 1: 569–600.

[5] Hoem, Jan M. and Erling Berge (1974 a): "Some problems in Hadwiger fertility graduation". Statistisk Sentralbyrå, ANO IO 74/5.

[6] Hoem, Jan M. and Erling Berge (1974 b): "Theoretical and empirical results on the analytic graduation of fertility curves". Side 2–10 i Hoem et al.: "Two papers on analytic graduation". Statistisk Sentralbyrå, ANO IO 74/17.

Some practical experiences with analytic graduation

by E. Berge and J. M. Hoem

In connection with a study of regional fertility in Norway, the authors have tried to investigate whether the Hadwiger function is systematically better or worse as a graduating function than a function proportional to the density of the gamma distribution, called the GG function here. The results do not provide anything like an unambiguous answer. In a large number of cases, the GG function fits the data better than the Hadwiger function. On the other hand, the latter seems to be the more flexible one, and there is a lot of evidence that it may be superior when the fertility curve is strongly skewed to the left and has a heavy upper tail.

Once a graduating function has been selected, one may get considerable variation in the estimates of its parameters as well as in the curves fitted by varying the (single year) age classes included in the graduation process. This is particularly noticeable when the graduation is carried out by means of chi-square. It is less important (in our data) in least squares graduation. The reason is that the former method gives more weight to age classes where fertility is small. Since this is the case for ages above age 45, say, the upper tail of the graduating function will be pressed down towards the abscissa axis more decisively the more of the higher age classes one includes in the graduation. Even though fertility is low at the very early ages too, no similar effect has been seen in that tail in the present data.

Norstedts Tryckeri, Stockholm 1975

Särtryck ur Statistisk tidskrift 1975: 5

The Demographic Interpretation of the Basic Parameters in Hadwiger Fertility Graduation

by Jan M. Hoem[1] *and Britta Holmbeck*[2]

1. Introduction and summary

Let us define a fertility curve as the diagram of a sequence of age-specific fertility rates plotted against age. When such a curve is graduated analytically, a nice, parametric function will be fitted to it. In the present paper we address ourselves to the question of how one can interpret the fitted values of the parameters of the graduating function. In order to be specific, we have chosen to concentrate on graduation by means of the Hadwiger function. Over the years, this function has been used to graduate a substantial number of fertility curves, and both of the present authors have been involved in projects where Hadwiger graduation has been carried out. In principle, the ideas presented here have general application, however, and they are not restricted to the Hadwiger function.

Our present point of departure, then, is the Hadwiger function with four parameters which in theory might be taken to represent the total fertility rate, the mean and modal ages of childbearing, and the variance of the age at childbearing, respectively. We show that except in the case of the modal age, this interpretation should be approached with considerable care. We define three *other* parameters who, in a real-life graduation, seem more natural candidates for the interpretation stated, and we suggest that the corresponding names be given to these "secondary" parameters. We also show that the values of the latter can differ appreciably from the "original" parameters, whose direct demographic interpretation consequently appears fuzzy. Since analytic graduation can be useful as a basis of fertility projections, we finally discuss the status of the latter in the light of our present findings.

A critical discussion of some completely different aspects of the concepts of mean and variance of the age at childbearing has been given previously by Hoem (1971). (In that connection, Jagers (1974) should be consulted.) The present argument should not be confused with that of the previous paper.

2. Hadwiger graduation

Let $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_x : x=a, a+1, ..., b\}$ be a sequence of observed age-specific fertility rates, calculated for single-year age groups, say. To carry out an analytic graduation of this sequence consists in selecting a nice, parametric function $g_x(\boldsymbol{\theta})$ and finding a value $\hat{\boldsymbol{\theta}}$ of the parameter vector $\boldsymbol{\theta}=(\theta_1, ..., \theta_r)$ such that the sequence

$$\mathbf{g}(\hat{\boldsymbol{\theta}}) = \{g_x(\hat{\boldsymbol{\theta}}) : x=a, a+1, ..., b\}$$

gives a good fit to the original sequence $\hat{\boldsymbol{\lambda}}$.

A possible choice of $g_x(\boldsymbol{\theta})$ is the value of the Hadwiger function at age x, i.e., $g_x = h_x$, with

$$h_x(\boldsymbol{\theta}) = \frac{RH}{T\sqrt{\pi}} \left(\frac{T}{x-D}\right)^{3/2} exp\left\{-H^2 \times \times \left(\frac{T}{x-D} + \frac{x-D}{T} - 2\right)\right\} \text{ for } x > D.$$

[1] Laboratory of Actuarial Mathematics, University of Copenhagen.
[2] Central Statistical Office of Denmark.

This parametrisation of the function is due essentially to Yntema (1969) and Gilje (1969). Hoem and Berge (1974a) have shown that it pays to reparametrise and take

$$\theta_1 = R, \theta_2 = M = D + T\left\{\left(1+\frac{16}{9}H^4\right)^{1/2} - 1\right\} / \left(\frac{4}{3}H^2\right), \theta_3 = A = T + D, \text{ and}$$

$$\theta_4 = S^2 = \tfrac{1}{2}T^2/H^2.$$

A systematic account of methods of analytic graduation has been given by Hoem (1972), and empirical applications to fertility curves have been presented by Hoem and Berge (1974) and their references. Further empirical results have just been published (Holmbeck, 1975).

Such investigations have proved $h_x(\boldsymbol{\theta})$ to be a flexible graduating function which will fit a wide range of fertility curves. The function does tend to have a rather heavy upper tail, i.e. it has a tendency to overestimate fertility in the uppermost fertile ages, particularly if it is fitted to rates for ages reaching up to the early forties only. This feature is more pronounced when the graduation is done by means of least squares than when the modified minimum chi-square method is applied (Hoem, 1972), simply because the latter puts more weight than the former on the ages of low fertility.

Even though in the general case the actual fitting of a function $g_x(\boldsymbol{\theta})$ is based on observed fertility rates for ages from a to b, inclusive, one may calculate the "fitted" value $g_x(\hat{\boldsymbol{\theta}})$ for other ages x also, of course, once the fitted value $\hat{\boldsymbol{\theta}}$ has been calculated. In this sense, the analytic graduation provides a means of extrapolating fertility beyond the ages used in the fitting procedure. (Cf. Berge and Hoem, 1974.)

To avoid the overestimation of fertility at the upper tail mentioned above one will frequently substitute zero fertility for the value of the Hadwiger function at the highest fertile ages, however, rather than extrapolate in the way just described.

3. Parameter interpretation

If we regard $h_x(\boldsymbol{\theta})$ as a function of a continuous x, then

$$R = \int_D^\infty h_x(\boldsymbol{\theta})dx,$$

and $h_x(\boldsymbol{\theta})/R$ is a probability density with mode M, mean A, and variance S^2. In fact, this is one reason why Hoem and Berge (1974a) selected this particular parametrisation.

We regard $h_x(\boldsymbol{\theta})$ as a gross maternity function or force of fertility (fertility intensity). In theory, therefore, one might interpret R as the corresponding total fertility rate, and similarly, M and A as the modal and mean ages of childbearing, and S^2 as the variance of the age at childbearing. When analytic graduation metods are applied to real-life fertility curves, such an interpretation should be treated with some care. It rests on a definition of $h_x(\boldsymbol{\theta})$ as a function of x for $D < x < \infty$, while in practice one will rarely be willing to accept $h_x(\boldsymbol{\theta})$ as representing a gross maternity function outside of a subinterval $<\alpha, \beta>$ corresponding to the fertile age span. The mode M of $h_x(\boldsymbol{\theta})$ will lie in $<\alpha, \beta>$, which essentially validates its demographic interpretation mentioned above. Unless $h_x(\boldsymbol{\theta})$ is very small outside of $<\alpha, \beta>$ this is not so for R, A, and S^2, however. Let us define

$$R_k = R_k(\boldsymbol{\theta}) = \int_\alpha^\beta x^k h_x(\boldsymbol{\theta})dx. \qquad (1)$$

Then the total fertility rate corresponding to $h_x(\boldsymbol{\theta})$ is R_0, not R. Similarly, the age at childbearing has a calculated mean of $a = R_1/R_0$ and a variance of

$$s^2 = s^2(\theta) = \frac{R_2}{R_0} - \left(\frac{R_1}{R_0}\right)^2,$$

not A and S^2, respectively. Evidently, R_0, a, and s^2 will be functions of R, M, A, and S^2, but they will not coincide with their respective counterparts.

At first sight, the distinction between (R, A, S^2) and (R_0, a, s^2) may seem academic, but this first impression does not stand up to empirical scrutiny, as we show in the next Section. Therefore, the distinction should be taken into account in practical work. Holmbeck (1975) seems to have been the only author to do so yet.

4. Empirical results

4.1. To investigate the practical importance of the difference between (R, A, S^2) and (R_0, a, s^2) we have selected seven Hadwiger curves which have appeared in the literature already. Each of these is characterized by a value of R, M, A, and S^2, and for each of them we have calculated R_0, a, and s^2 by numerical integration, as follows.

We have let $\alpha = 15$, and have calculated values of R_0 etc. for $\beta = 45$ and $\beta = 50$, separately. The calculations have been carried out on the University of Copenhagen UNIVAC 1110 computer by means of a standard program called SIM3NI from its math-pack (UNIVAC, 1970). This program is based on Simpson's 3/8 rule, and the numerical results appear in Table 1 in lines denoted "Simpson 3/8". (In one case, the iteration process of the program did not converge.)

4.2. Evidently, the choice of numerical integration method determines the accuracy of our approximation to the integral in (1). Something less sophisticated than the standard program SIM3NI consists in letting

$$\Sigma_k = \Sigma_k(\theta) = \sum_{x=\alpha'}^{\beta'} x^k h_x(\theta)$$

and using Σ_k as an approximation to R_k. Here α' and β' are integers suitably close to α and β, respectively. For comparison, we have used this method too, with $\alpha' = \alpha$ and $\beta' = \beta - 1$ (for our one choice of value for α and both choices for β). Let us call this the *simple summation method.* It consists of course in behaving as if fertility were defined for single year age groups rather than for a continuous age variable. The numerical results appear in Table 1 in lines denoted "Simple sum". In our opinion, *the approximation obtained by the simple summation method to the more accurate values given by Simpson's 3/8 method in the cases studied must be quite satisfactory for most purposes in demography.* This is our first main conclusion.

4.3. These seven curves have been chosen so as to provide a fairly wide range of levels and age structures of fertility. Among other things, we have wanted curves with a heavy upper tail as well as curves with a light upper tail. The evaluation of how heavy an upper tail a curve has, can be made subjectively by inspection of its diagram, or more objectively by the calculation of one or more measures such as the variance, the normalized tail area $1 - R_0/R$ for suitable α and β, or simply some normalized ordinate like $h_{45}(\theta)/R$. In Table 1, our seven curves have been arranged by increasing level of the normalized tail area $1 - R_0/R$ with $\alpha = 15$, $\beta = 45$. (Other measures may result in minor changes in the ranking.) The intention is to get to curves with a progressively heavier upper tail as one moves downwards in the table.

4.4. Due to the form of the Hadwiger curve, described in Section 2 above, one would expect to find that $R > R_0$, $A > a$, and $S^2 > s^2$.

Table 1. Parameter values for seven fertility curves

Curve	Parameter type	Age span	Area	Mean	Variance	Main source[1]	Subsidiary sources
Växjö,[9] 1968–73	Hadwiger [2,5]		2.056	27.271	30.875	Holmbeck (1975)[1]	
	Simpson 3/8 [3,6]	15–45	2.04	27.17	28.45		
	Simple sum[4,6]	15–44	2.04	27.15	28.27		
	Simpson 3/8[3,6]	15–50	2.05	27.26	30.02		mode=25.411
	Simple sum[4,6]	15–49	2.05	27.24	30.00		tail[7]=6.5
Oslo, 1968–71	Hadwiger		1.880	27.280	32.460	Hoem and Berge (1974b)[1]	Berge (1974), Berge and Hoem (1974)[1]
	Simpson 3/8	15–45	1.87	27.15	29.42		
	Simple sum	15–44	1.86	27.12	29.20		
	Simpson 3/8	15–50	1.88	27.25	31.32		mode=25.230
	Simple sum	15–49	1.88	27.24	31.27		tail=7.8
Hungary, 1961	Hadwiger		1.963	25.695	42.423	Gilje (1969)[1]	Hoem and Berge (1974a)
	Simpson 3/8	15–45	1.93	25.30	32.35		
	Simple sum	15–44	1.93	25.27	31.87		
	Simpson 3/8	15–50	1.95	25.49	(non-convergence)[8]		mode=21.628
	Simple sum	15–49	1.95	25.48	36.03		tail=15.6
Berg, Härjedalen, and Åre[9], 1968–73	Hadwiger		1.948	28.062	42.495	Holmbeck (1975)[1]	
	Simpson 3/8	15–45	1.91	27.67	34.15		
	Simple sum	15–44	1.91	27.63	33.66		
	Simpson 3/8	15–50	1.94	27.91	38.29		mode=25.122
	Simple sum	15–49	1.93	27.89	38.03		tail=18.7
Sunndal, Meråker, Verran, and Rana,[10] 1968–71	Hadwiger		2.824	26.700	47.180	Berge (1974)[1]	
	Simpson 3/8	15–45	2.76	26.16	34.03		
	Simple sum	15–44	2.76	26.13	33.44		
	Simpson 3/8	15–50	2.80	26.42	38.96		mode=22.470
	Simple sum	15–49	2.80	26.40	38.55		tail=21.6
Norway, 1966	Hadwiger		2.957	27.502	53.453	Hoem and Berge (1974a)[1]	Gilje (1969)[1]
	Simpson 3/8	15–45	2.87	26.89	38.69		
	Simple sum	15–44	2.87	26.83	38.12		
	Simpson 3/8	15–50	2.92	27.21	44.47		mode=23.628
	Simple sum	15–49	2.92	27.17	44.12		tail=28.5
13 communes in Western Norway, 1968–71	Hadwiger		3.500	28.440	54.000	Hoem and Berge (1974b)[1]	Hoem and Berge (1974a)[1]
	Simpson 3/8	15–45	3.38	27.65	35.90		
	Simple sum	15–44	3.37	27.60	35.16		
	Simpson 3/8	15–50	3.45	28.02	42.29		mode=24.120
	Simple sum	15–49	3.44	27.99	41.75		tail=33.5

[1] Contains diagram of curve.
[2] Area $=R$, mode $=M$, mean $=A$, variance $=S^2$.
[3] Area $=R_0$, mean $=a$, variance $=s^2$.
[4] Area $=\Sigma_0$, mean $=\Sigma_1/\Sigma_0$, variance $=\Sigma_2/\Sigma_0-(\Sigma_1/\Sigma_0)^2$.
[5] Three input decimals.
[6] Rounded from eight decimals.
[7] Tail $=1000\,(1-R_0/R)$ for $\alpha=15$, $\beta=45$.
[8] The numerical integration process did not converge in this case.
[9] Swedish communes.
[10] Norwegian communes.

Since the calculation of A puts greater weight on high ages than the calculation of R does, one would expect $A-a$ to be larger than $R-R_0$. For the same kind of reason, one would expect to get $S^2-s^2>A-a$. One would also expect the differences $R-R_0$, $A-a$, and S^2-s^2 largely to increase as the upper tail of the curve becomes heavier, i.e.,

Table 2. Parameter value differences for the seven fertility curves

Curve	Age span	Difference in area, $R-R_0$	Difference in mean, $A-a$	Difference in variance, S^2-s^2
Växjö, 1968–73	15–45	0.013	0.10	2.43
	15–50	0.004	0.01	0.86
Oslo, 1968–71	15–45	0.015	0.13	3.04
	15–50	0.005	0.03	1.14
Hungary, 1961	15–45	0.031	0.40	10.07
	15–50	0.013	0.21	6.39[1]
Berg, Härjedalen,	15–45	0.036	0.39	8.35
Åre, 1968–73	15–50	0.013	0.15	4.21
Sunndal etc., 1968–71	15–45	0.061	0.54	13.15
	15–50	0.027	0.28	8.22
Norway, 1966	15–45	0.084	0.61	14.76
	15–50	0.038	0.29	8.98
13 Norwegian communes, 1968–71	15–45	0.117	0.79	18.10
	15–50	0.053	0.42	11.71

[1] This item is $S^2-\{\Sigma_2/\Sigma_0-(\Sigma_1/\Sigma_0)^2\}$.

as one moves downwards in Table 1. Finally, one would expect the discrepancies to be larger when $\alpha=15$, $\beta=45$ than when $\alpha=15$, $\beta=50$, since we include a larger section of the domain of the Hadwiger function in the latter case.

In order to check these expectations, we have listed the differences calculated in Table 2. An inspection of the table shows that they are all well confirmed in our data.

4.5. It is gratifying to have one's expectations confirmed as we did above, of course, but what Table 2 tells us about the numerical values of the differences $R-R_0$, $A-a$, and S^2-s^2 is more important. Our main conclusions are as follows.

(i) For $\alpha=15$, $\beta=50$, *the difference $R-R_0$ is very small, probably negligible.* This is also the case even if $\alpha=15$, $\beta=45$, though in this case the difference does get up to a value of almost 3.5 % of R_0 when the tail is particularly heavy.

(ii) Even for $\alpha=15$, $\beta=50$, the difference $A-a$ can be noticeable. In our data it gets up to a value as high as 0.4, and even to 0.8 when $\alpha=15$, $\beta=45$. *For curves with anything over very light upper tails,* A *may exceed* a *by an amount probably too large to make* A *an acceptable approximation to* a *for many purposes,* particularly when $\beta=45$.

(iii) For S^2-s^2, a conclusion similar to the one in (ii) holds, only somewhat more strongly formulated. *For curves whose upper tails are not very light, S^2-s^2 seems unacceptably large* for many purposes, even when $\alpha=15$, $\beta=50$.

5. Fertility projections

The result of the analytic graduation of fertility curves can be useful in many circumstances, such as in connection with fertility projections. [Mitra and Romaniuk (1973) and Romaniuk (1973) stress this point.] Let us assume that one has established $g_x(\boldsymbol{\theta})$ as a suitable graduating function for a particular population, and that fitted values $\hat{\boldsymbol{\theta}}(t)$ have been calculated for $\boldsymbol{\theta}$ for each of a number of past time periods $t=T_0$, T_0+1, ..., T_1. On the basis of these observations, one may perhaps project future values $\boldsymbol{\theta}^*(t)$ for $\boldsymbol{\theta}$ for periods $t=T_2$, T_2+1, ..., T_3. The introduction of these future values into $g_x(\boldsymbol{\theta})$ will then give projected future fertility schemes

$\boldsymbol{\lambda}^*(t)=\{\lambda^*_x(t) : x=\alpha', \alpha'+1, ..., \beta'\}$
for $t=T_2$, T_2+1, ..., T_3, with
$\lambda^*_x(t)=g_x\{\boldsymbol{\theta}^*(t)\}$.

A proper demographic interpretation of the elements θ_i in the parameter vector $\boldsymbol{\theta}=(\theta_1, ..., \theta_r)$ would be most helpful in the process of selecting future values $\boldsymbol{\theta}^*(t)$ for $\boldsymbol{\theta}$. If $g_x(\boldsymbol{\theta})$ is the Hadwiger function, therefore, it would be nice if we could reparametrise it once more and use R_0, a and s^2 as parameters instead of R, A and S^2. (The mode M would be carried over to the new parametrisation.)

To do so in practice, one would need to express R, A and S^2 as reasonably nice functions of R_0, a, s^2 and M. Unfortunately, we do not know how this can be done in general. Until further relations between the parameter sets have been established, therefore, it looks as if we are stuck with using (R, M, A, S^2) as our basic parameter vector in connection with fertility projections based on the Hadwiger function, even though the demographic interpretation of A and S^2 (and to some extent R also) remains fuzzy. To develop something better is a subject of future research. The matter cannot be solved by simply substituting future values of a and s^2 (and R_0) into the Hadwiger function instead of corresponding values of A and S^2 (and R).

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