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TWO NOTES ON THE STOCHASTIC SPECIFICATION OF A COMPLETE SET OF CONSUMER DEMAND FUNCTIONS

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A. REMARKS ON SOME GENERAL RESTRICTIONS ON THE DISTRIBUTION OF THE DISTURBANCES OF THE CONSUMER DEMAND FUNCTIONS

1.

This note deals with the structure of the random disturbances in a complete system of demand functions for consumption commodities. More specifically, the purpose is to discuss restrictions which the probability distribution of the disturbances must satisfy in order to ensure that the complete equation system is consistent with utility maximisation when total consumption expenditure is exogenously given.

Consider the following system of demand functions:

$$(1) \qquad x_{it} = f_i(y_t, p_{1t}, \dots, p_{Nt}) + v_{it},$$

where x_{it} and p_{it} denote the quantity demanded and the price respectively of commodity no. i in period no. t ($i = 1,\dots, N$; $t = 1,\dots, T$), y_t total consumption expenditure in period no. t, definitionally equal to the sum of the values of the N commodities, i.e.,

$$(2) \qquad y_t = \sum_{i=1}^{N} p_{it}x_{it} \qquad (t = 1,\dots, T),$$

and v_{it} a stochastic disturbance term. The unspecified functions $f_1,\dots, f_N$ are supposed to satisfy conditions which conform to utility maximising behaviour. In particular, this implies that the 'adding-up condition'

$$(3) \qquad \sum_i p_{it}f_i(y_t, p_{1t},\dots, p_{Nt}) = y_t,$$

and the 'homogeneity conditions'

$$(4) \qquad f_i(y_t, p_{1t},\dots, p_{Nt}) = f_i(ky_t, kp_{1t},\dots, kp_{Nt}) \qquad (i = 1,\dots, N)$$

hold identically in y_t, $p_{1t},\dots, p_{Nt}$, and k.

2.

Equations (1)-(3) involve the following restrictions on the disturbances:

(5) $\sum_i p_{it} v_{it} = 0$ (t = 1,..., T).

In the sequel, we shall consider the p's and y as non-stochastic variables and assume (i) that all disturbances have zero expectations, and (ii) that disturbances relating to different observations (periods) are uncorrelated, i.e.,

(6) $E(v_{it}) = 0,$

(7) $E(v_{it} v_{js}) = \begin{cases} \sigma_{ijt} & \text{for } s = t \\ 0 & \text{for } s \neq t \end{cases}$

(i, j = 1,..., N)
(t, s = 1,..., T).

Multiplying eqs. (5) by v_{jt} and taking expectations we obtain

(8) $\sum_i p_{it} \sigma_{ijt} = 0$ (j = 1,..., N) (t = 1,..., T).

Since $\sigma_{ijt} = \sigma_{jit}$ for all i, j, and t, (8) may alternatively be written as

(8a) $\sum_j p_{jt} \sigma_{ijt} = 0$ (i = 1,..., N) (t = 1,..., T).

From this we derive:

Proposition 1. (i) The variances/covariances of the disturbances of the demand functions cannot take the same values for all observations ($\sigma_{ijt} = \sigma_{ij}$ for all i, j, and t) unless the prices are equal across observations (periods) up to a common factor of proportionality. The necessary and sufficient conditions for constant variances / covariances can be written, formally, as (a) $p_{it} = k_t p_{i1}$ for all i and t (k_t is a positive constant), and (b) $\Sigma_i p_{i1} \sigma_{ij} = 0$ for all j.

(ii) Subsets of the variances/covariances of the disturbances can be specified to have the same distribution for all observations. We may, for instance, restrict the variances/covariances relating to the first N-1 commodities to be independent of t, i.e. $\sigma_{ijt} = \sigma_{ij}$ for i, j = 1,..., N-1, provided we accept that $\sigma_{Njt} = -(1/p_{Nt}) \Sigma_i p_{it} \sigma_{ij}$ (j = 1,..., N).

3.

Multiplying both sides of the equality sign of eq. (1) by p_{it} gives the set of 'expenditure functions', with disturbances

$$(9) \qquad w_{it} = p_{it}v_{it} \qquad (i = 1,\ldots, N) \quad (t = 1,\ldots, T),$$

which satisfy the restrictions

$$(10) \qquad \sum_i w_{it} = 0 \qquad (t = 1,\ldots, T).$$

(Compare eqs. (5).)

Hence, we may specify the vectors of disturbances of the expenditure functions $(w_{1t},\ldots, w_{Nt})$ to be identically distributed for all observations without violating the 'adding-up conditions' (3) and (5). More explicitly, we may assume

$$(11) \qquad E(w_{it}) = 0$$

$$(12) \qquad E(w_{it}w_{js}) = \delta_{ts}\mu_{ij} \qquad \left\{ \begin{array}{l} (i, j = 1,\ldots, N) \\ (t, s = 1,\ldots, T), \end{array} \right.$$

where $\delta_{ts} = 1$ for $s = t$, and 0 otherwise, provided

$$(13) \qquad \sum_i \mu_{ij} = 0 \qquad (j = 1,\ldots, N).$$

If we accept this specification, we get the following expressions for the variances/covariances of the disturbances of the demand functions (compare eqs. (7), (9), and (12)):

$$(14) \qquad \sigma_{ijt} = \frac{\mu_{ij}}{p_{it}p_{jt}} \qquad (i, j = 1,\ldots, N) \quad (t = 1,\ldots, T).$$

Conditions (13) imply that the variance/covariance matrix of the w_{it}'s is singular: consequently, $w_{1t},\ldots, w_{Nt}$ cannot be restricted to be mutually uncorrelated. Since the variance μ_{ii} is positive, at least one of the covariances μ_{ij} must be negative. Thus, we have:

<u>Proposition 2.</u> (i) The variances/covariances of the disturbances of the expenditure functions can be specified to take the same values for all observations (μ_{ij} independent of t).

(ii) The variance/covariance matrix is singular; an assumption that all disturbances are mutually uncorrelated is inconsistent with the specification (1)-(3).

The latter result is well-known from e.g. the literature dealing with the Stone linear expenditure system.[1] One way proposed to solve the problem of singularity is to delete one commodity group from consideration and assume that the disturbances relating to the remaining ones are uncorrelated. This approach is rather unsatisfactory, however, as the commodity group to be deleted has to be chosen arbitrarily, and the choice may seriously affect the estimates of the coefficients of the model.

4.

The homogeneity conditions (4) restrict the admissible set of demand functions. It seems reasonable to place similar restrictions on the random components of demand, i.e., to require that a proportional change of all prices and total expenditure leave the distribution of the disturbances of the demand functions, or at least its second order moments, unaffected. Would this requirement be compatible with the specification (11)-(13)? Obviously, the answer is no. If all the μ_{ij}'s are constants, multiplying all prices and total expenditure by a factor k reduces the variances/co-variances of the demand function disturbances by a factor of $1/k^2$. (Compare eq. (14).) For this reason, the specification (11)-(13) is not fully satisfactory from a theoretical point of view, although it has been extensively used in practice, e.g. in connection with the Stone linear expenditure system.

5.

This motivates considering the following, more general, problem: Assume that both sides of the equality sign of eq. (1) are multiplied by a non-stochastic weight b_{it}, its value depending on the commodity group as well as on the number (period) of observation. Which restrictions should be imposed on the b_{it}'s to ensure that the adding-up conditions as well as the homogeneity conditions are satisfied?

The transformed disturbances are

1) See e.g., Pollak and Wales [2], p. 615.

(15) $$u_{it} = b_{it}v_{it} \qquad (i = 1,\ldots, N) \quad (t = 1,\ldots, T),$$

with second order moments equal to

(16) $$\eta_{ijt} \equiv E(u_{it}u_{jt}) = b_{it}b_{jt}\sigma_{ijt},$$

where σ_{ijt} is defined as in eq. (7). For the adding-up conditions to hold, eq. (8) must be satisfied. The corresponding restrictions expressed in terms of the η_{ijt}'s are

(17) $$\sum_i \frac{p_{it}}{b_{it}} \eta_{ijt} = 0 \qquad (j = 1,\ldots, N) \quad (t = 1,\ldots, T).$$

Let us now restrict the second order moments of the u_{it}'s to take the same values for all observations, i.e. $\eta_{ijt} = \eta_{ij}$ for all i, j, and t. For (17) to be satisfied identically in the p_{it}'s when the latter condition is imposed, the weights must have the form

(18) $$b_{it} = \frac{p_{it}}{c_i d_t} \qquad (i = 1,\ldots, N) \quad (t = 1,\ldots, T),$$

where the c_i's are subject to the restrictions

(19) $$\sum_i c_i \eta_{ij} = 0 \qquad (j = 1,\ldots, N).$$

The d_t's, however, can be chosen arbitrarily.

Using this specification, the variances/covariances of the disturbances of the demand functions can be written in the following way:

(20) $$\sigma_{ijt} = \frac{\eta_{ij}}{b_{it}b_{jt}} = \frac{c_i c_j d_t^2}{p_{it}p_{jt}} \eta_{ij} \qquad (i, j = 1,\ldots, N) \quad (t = 1,\ldots, T).$$

What about the homogeneity condition in this case? From (20) we see that it is perfectly possible to specify η_{ij} to be constant, and at the same time pay regard to the requirement that all the σ_{ijt}'s be unaffected when all prices and total expenditure change proportionally. We only have to let d_t be a function homogeneous of the first degree in prices and total consumption expenditure.

Summing up, we have:

Proposition 3. (Generalization of Proportions 1 and 2.)

(i) If we transform the demand functions (1) by multiplying by the factor b_{it}, constraining the transformed disturbances to have the same variance/covariance matrix (η_{ij}) for all observations, then b_{it} must be of the form $b_{it} = p_{it}/(c_i d_t)$ for all i and t, where $\Sigma_i \eta_{ij} c_i = 0$ for all j.

(ii) If we impose the additional requirement of homogeneity, i.e., that a proportional change of all prices and total expenditure leave the second order moments of the demand function disturbances v_{it} unaffected, then d_t must be homogeneous of degree one in $p_{1t}, \ldots, p_{Nt}$ and y_t. I.e., the weights b_{it} must be of the form $b_{it} = p_{it}/(c_i \lambda(p_{1t}, \ldots, p_{Nt}, y_t))$ for all i and t, where $\Sigma_i c_i \eta_{ij} = 0$ for all j, and λ is homogeneous of degree one.

6.

It is illuminating to consider some applications of the results summarised in Proposition 3.

Owing to an idea that the scope for variations in consumption habits is larger for higher levels of consumption (welfare) than for lower ones, the disturbances of the demand functions are often supposed to be heteroscedastic. One version of this hypothesis might be that the standard error of v_{it} in the 'original' demand function (1) is proportional to total consumption expenditure y_t. Another version might be that the standard error of v_{it} is proportional to expected consumption $Ex_{it} = f_i\ (y_t, p_{1t}, \ldots, p_{Nt})$. This corresponds to fixing the weights b_{it} equal to $1/y_t$, and $1/Ex_{it}$, respectively, while restricting the resulting u_{it}'s to be homoscedastic. Would this be consistent with the basic assumptions of our model? Generally, the answer is no in both cases, as is readily seen from eq. (18). If b_{it} were equal to $1/y_t$, then we should have $p_{it}/c_i = d_t/y_t$. This equality could hold only if all prices change proportionally. On the other side, the restriction $b_{it}=1/Ex_{it}$ would imply $p_{it}Ex_{it}=p_{it}f_i(\cdot)=c_i d_t$. Combining this with eq. (3) we find $f_i(\cdot) = (c_i/\Sigma_k c_k)y_t/p_{it}$; i.e., the specification would be admissible for demand functions giving constant budget proportions only.

A third example is to let $b_{it}=p_{it}/y_t$, i.e., to apply the following set of "income normalized expenditure functions" or "budget proportion functions":

$$(21)\qquad \frac{p_{it}}{y_t} x_{it} = \frac{p_{it}}{y_t} f_i\ (y_t, p_{1t}, \ldots, p_{Nt}) + u_{it}^{(1)},$$

where

$$(22) \qquad u_{it}^{(1)} = p_{it}v_{it}/y_t.$$

This specification satisfies the adding-up as well as the homogeneity restrictions, since it corresponds to eq. (18) with $c_i = 1$ and $d_t = y_t$, the latter being homogeneous of the first degree in total expenditure and prices. Moreover, it pays regard to the idea of heteroscedasticity: assuming that the standard error of $u_{it}^{(1)}$ has the same value for all t implies that the standard error of v_{it} is proportional to y_t. This kind of transformation forms the basis of the so-called "Rotterdam model". The Rotterdam model, however, applies this transformation to the demand functions written in terms of first-differences, assuming that the corresponding first-differenced disturbances are non-autocorrelated. (For details, see e.g. Barten [1], and Theil [3], Ch. 2.)

A fourth, and final, example is to let $b_{it} = p_{it}/P_t$, where P_t is a price index homogeneous of the first degree in prices. This specification satisfies the adding-up and the homogeneity restrictions, but it pays no regard to the idea of heteroscedasticity.

R E F E R E N C E S

[1] Barten, A.P.: Evidence on the Slutsky Conditions for Demand Equations. Review of Economics and Statistics, vol. 49 (1967), pp. 77-84.

[2] Pollak, R.A. and Wales, T.J.: Estimation of the Linear Expenditure System. Econometrica, vol. 37 (1969), pp. 611-628.

[3] Theil, H.: Theory and Measurement of Consumer Demand, Vol. 1. (Amsterdam: North-Holland Publishing Company, 1975.)

B. STOCHASTIC SPECIFICATION OF CONSUMER DEMAND FUNCTIONS WHEN INTRODUCING STOCHASTIC ELEMENTS INTO THE UTILITY FUNCTION AND THE FIRST-ORDER CONDITIONS: AN EXAMPLE BASED ON THE STONE-GEARY UTILITY FUNCTION

1.

The specification of the stochastic elements of a complete system of consumer demand functions is an interesting and important problem in applied econometrics. However, some aspects of the problem seem to be undeservedly neglected in the literature. The strategy commonly chosen is a two-stage procedure; first, to specify a set of non-stochastic (i.e., exact) demand functions which conform to utility-maximising behaviour, and second, to furnish these functions with (additive) stochastic disturbances.[1)] Rarely, attempts are made to connect the two parts of the model formulation.[2)]

On the other hand, when dealing with a formally similar problem within the context of producer's behaviour - i.e., when constructing the product supply and factor demand functions of a (typical) profit-maximizing firm with a parametrically specified production function - the standard approach is essentially different. The stochastic elements are introduced into the model from the outset, in the form of disturbances in the production function and in the equations representing the first-order conditions for maximising "average"(or "expected") profit.[3)] From this structural specification the reduced form equations, i.e., the product supply and factor demand equations, and the restrictions to be placed on their stochastic elements can be derived.

A pertinent question is: Why not follow the latter approach also when specifying the stochastic structure of the consumer demand functions? One answer may be that the quantity produced is an observable variable, whereas its counterpart within the consumer demand framework, the utility level, is not; consequently, the utility function cannot be considered a structural equation in the strict sense. An alternative (but related) way of explaining the current practice is to refer to the fact that the econometrician frequently is interested in the properties of the production function without being particularly interested in the product supply or factor demand functions, while the utility function is of limited practical interest by itself. Neither of these answers, however, is entirely satisfactory.

1) See e.g. Deaton [1], Chs. 3 and 4.
2) However, see Theil [7], Ch. 2.6, for an interesting exception.
3) See e.g. Marschak and Andrews [4], and Nerlove [5].

The purpose of this note is to give an example of a set of consumer demand functions derived when the stochastic elements are introduced into the utility function and the equations representing the first order conditions for (constrained) utility maximisation. We shall assume that the "average" utility function belongs to the Stone-Geary (Klein-Rubin) class. Otherwise, the specification is fairly general: Saving is introduced as a separate argument ("commodity") into the utility function, and various types of stochastic elements (disturbances, errors) are specified.

Suppose the utility function has the form

$$(1) \qquad U = \sum_{i=1}^{N} \beta_i^* \log (x_i^* - \gamma_i^*) + \beta^* \log (s^* - \gamma^*),$$

where x_i^* and s^* denote the quantity of the i'th commodity consumed and the volume of saving respectively ($i = 1, \dots, N$). The coefficients β_i^* γ_i^*, β^*, and γ^* are supposed to be known by the consumer when carrying out his optimisation, but are, of course, unknown to the econometrician. Moreover, we shall assume that the coefficients differ between consumers (or, more generally, units of observation), and that the differences appear to the econometrician as resulting from random variations, i.e.,

$$(2) \qquad \begin{cases} \beta_i^* = \beta_i + \varepsilon_i & (i = 1, \dots, N), \\ \beta^* = \beta + \varepsilon, \end{cases}$$

$$(3) \qquad \begin{cases} \gamma_i^* = \gamma_i + v_i & (i = 1, \dots, N), \\ \gamma^* = \gamma + v, \end{cases}$$

where β_i, β, γ_i, and γ denote the average (non-stochastic) coefficients, and ε_i, ε, v_i, and v are stochastic errors. Finally, in order to keep the specification of the model fairly general, we assume that the values of x_i^* and s^* observed by the econometrician (e.g., the values reported by the consumer) differ from their "true" values by stochastic errors u_i and u; i.e., the values observed are

$$(4) \qquad x_i = x_i^* + u_i \qquad (i = 1, \dots, N),$$

$$(5) \qquad s = s^* + u.$$

Let p_i and P denote the price of the i'th commodity and the "price" of saving (i.e., the price index used to deflate nominal saving to get its real value), respectively. We suppose that these variables can be observed without error. Let further y and y^* denote the income observed by the econometrician and the "true" income (i.e., the income known to the consumer, but unknown to the econometrician), respectively. We then have

$$(6) \qquad y^* = \sum_i p_i x_i^* + Ps^*,$$

$$(7) \qquad y = \sum_i p_i x_i + Ps.$$

From eqs. (4) - (7) follows

$$(8) \qquad y = y^* + \sum_i p_i u_i + Pu.$$

The problem of optimisation as regarded from the consumer's point of view is the following: Maximise the utility level U with respect to x_i^* (i = 1,..., N) and s^*, subject to the budget constraint (6), taking p_i (i = 1,..., N), P (a function of the p_i's), and y^* as given. The first-order marginal conditions can be written as

$$(9) \qquad \frac{\partial U}{\partial x_i^*} = \frac{\beta_i^*}{x_i^* - \gamma_i^*} = \frac{\omega p_i}{1+w_i} \qquad (i = 1,\ldots, N),$$

$$(10) \qquad \frac{\partial U}{\partial s^*} = \frac{\beta^*}{s^* - \gamma^*} = \frac{\omega P}{1+w},$$

where ω denotes the marginal utility of income, and w_i and w are random disturbances intended to take care of errors in maximisation. We may, for instance, imagine that the consumer, for one reason or another, is unable to attain the maximising utility level exactly, or alternatively, that his target can be described only approximately as constrained maximisation of the utility function (1).

3.

From eqs. (6), (9) and (10) we get, after elimination of ω, the following system of 'expenditure functions':

$$(11) \qquad p_i x_i^* = p_i \gamma_i^* + \frac{\beta_i^*(1+w_i)}{\sum \beta_j^*(1+w_j) + \beta^*(1+w)} \left[y^* - \sum p_j \gamma_j^* - P\gamma^*\right] \quad (i=1,\ldots,N),$$

and the following 'saving function':

$$\text{(12)} \qquad Ps^* = P\gamma^* + \frac{\beta^*(1+w)}{\Sigma\beta_j^*(1+w_j) + \beta^*(1+w)} [y^* - \Sigma p_j\gamma_j^* - P\gamma^*].$$

The left hand sides of eqs. (11) and (12) represent the "true" expenditure spent on the i'th commodity, and the "true" value of saving, respectively. The first terms on the right hand side of the equations represent the corresponding values of 'minimum consumption' and 'minimum saving' respectively, while the expression in the square brackets may be interpreted as the "true" value of the 'supernumerary income'. Finally, the fractional expressions before the square brackets represent the marginal propensity to consume of the i'th commodity and the marginal propensity to save, respectively. The total propensity to consume is $\Sigma\beta_i^*(1+w_i)/\{\Sigma\beta_j^*(1+w_j) + \beta^*(1+w)\}$. Owing to eqs. (2) and (3), all the variables and coefficients mentioned above are random variables.

The marginal propensities to consume and to save can be decomposed into a non-stochastic and a stochastic part. By using eqs. (2) we have

$$\text{(13)} \qquad \frac{\beta_i^*(1+w_i)}{\Sigma\beta_j^*(1+w_j) + \beta^*(1+w)} = \alpha_i + \delta_i \qquad (i = 1,\ldots, N),$$

$$\text{(14)} \qquad \frac{\beta^*(1+w)}{\Sigma\beta_j^*(1+w_j) + \beta^*(1+w)} = \alpha + \delta,$$

where

$$\text{(15)} \qquad \alpha_i = \beta_i/\{\Sigma\beta_i + \beta\} \qquad (i = 1,\ldots, N)$$

is the "average" value of the marginal propensity to consume of the i'th commodity,

$$\text{(16)} \qquad \alpha = \beta/\{\Sigma\beta_i + \beta\}$$

is the "average" value of the marginal propensity to save, and

$$(17) \qquad \delta_i = \frac{(\Sigma\beta_j + \beta)(\varepsilon_i + \beta_i w_i + \varepsilon_i w_i) - \beta_i(\Sigma\varepsilon_j + \varepsilon + \Sigma\beta_j w_j + \beta w + \Sigma\varepsilon_j w_j + \varepsilon w)}{(\Sigma\beta_j + \beta)(\Sigma(\beta_j + \varepsilon_j)(1+w_j) + (\beta+\varepsilon)(1+w))} \qquad (i=1,\ldots,N),$$

$$(18) \qquad \delta = \frac{(\Sigma\beta_j + \beta)(\varepsilon + \beta w + \varepsilon w) - \beta(\Sigma\varepsilon_j + \varepsilon + \Sigma\beta_j w_j + \beta w + \Sigma\varepsilon_j w_j + \varepsilon w)}{(\Sigma\beta_j + \beta)(\Sigma(\beta_j + \varepsilon_j)(1+w_j) + (\beta+\varepsilon)(1+w))}$$

are stochastic errors. It is easily seen that $\Sigma\alpha_i + \alpha = 1$, and $\Sigma\delta_i + \delta = 0$.

By using eqs. (2), (3), (4), (8), (13), and (14) to eliminate the starred variables and coefficients in eqs. (11) and (12), the expenditure and saving functions can be written as

$$(19) \qquad p_i x_i = p_i \gamma_i + \alpha_i(y - \Sigma p_j \gamma_j - P\gamma) + p_i(v_i + u_i) - \alpha_i(\Sigma p_j(v_j + u_j) + P(u + v)) + \delta_i(y - \Sigma p_j \gamma_j - P\gamma) - \delta_i(\Sigma p_j(v_j + u_j) + P(u + v)) \qquad (i=1,\ldots, N).$$

$$(20) \qquad Ps = P\gamma + \alpha(y - \Sigma p_j \gamma_j - P\gamma) + P(u + v) - \alpha(\Sigma p_j(v_j + u_j) + P(u + v)) + \delta(y - \Sigma p_j \gamma_j - P\gamma) - \delta(\Sigma p_j(v_j + u_j) + P(u + v)),$$

or

$$(19a) \qquad p_i x_i = p_i \gamma_i + \alpha_i(y - \Sigma p_j \gamma_j - P\gamma) + U_i \qquad (i = 1,\ldots, N),$$

$$(20a) \qquad Ps = P\gamma + \alpha(y - \Sigma p_j \gamma_j - P\gamma) + U,$$

for short. It is readily observed that the composite error terms U_i and U have the property $\Sigma U_i + U = 0$ regardless of the assumptions made with respect to the distribution of the errors and disturbances u_i, v_i, ε_i, w_i, u, v, ε, and w of the structural form of the model. Our approach automatically ensures that the adding-up condition is satisfied.[4)]

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4) Compare Pollak and Wales [6], whose modification of the Stone model proposed on pp. 613-614 may be considered a special case of our model.

4.

The consumption function associated with the expenditure and saving functions (19)-(20) can be easily derived.
Defining

(21) $c = \Sigma p_i x_i = y - Ps,$

eq. (20) yields

(22) $$c = (1 - \alpha)(y - P\gamma) + \alpha\Sigma p_j\gamma_j - P(u + v) + \alpha(\Sigma p_j(v_j + u_j) + P(u + v))$$
$$-\delta(y - \Sigma p_j\gamma_j - P\gamma) + \delta(\Sigma p_j(v_j + u_j) + P(u + v))$$
$$= (1 - \alpha)(y - P\gamma) + \alpha\Sigma p_j\gamma_j - U.$$

Here, α may be interpreted as the ("average") marginal propensity to save, and, correspondingly, $(1 - \alpha)$ represent the ("average") marginal propensity to consume.[5)]

By elimination of income y from eqs. (19a) and (22) the expenditure functions take the form

(23) $$p_i x_i = p_i\gamma_i + \frac{\alpha_i}{1-\alpha}(c - \Sigma p_j\gamma_j) + \frac{\alpha_i}{1-\alpha} U + U_i \quad (i = 1, \ldots, N),$$

which corresponds with the Stone LES system. We find, not surprisingly, that eqs. (23) satisfy the adding-up condition $\Sigma p_i x_i = c$ identically, owing to the fact that $\Sigma\alpha_i = 1 - \alpha$ and that $\Sigma U_i = -U$.

5.

So far, no assumptions have been made with respect to the probability distribution of the errors and disturbances of the model. Some remarks are in order.

First, a reasonable assumption is that the errors of observation in the quantities consumed and in the volume of saving have zero expectations and are uncorrelated with their true values, i.e.,

5) It is interesting to notice the formal similarity between eqs. (22) and (19a) on the one hand and the consumption function and the expenditure functions derived from the ELES approach, as suggested by Lluch, on the other. (See Lluch [2], and Lluch and Williams [3].) If 'minimum saving' is restricted to zero, i.e., $\gamma = 0$, the structural parts of the equations have in fact identically the same form. The stochastic specification of the ELES system, however, is different from ours. Compare Sec. 5 below.

(24) $Eu_i = Eu = 0$ for all i,

(25) $E(u_i x_j^*) = E(u_i s^*) = E(u x_j^*) = E(u s^*) = 0$ for all i, j.

By application of eqs. (6) and (8), this yields

(26) $Ey = Ey^*$,

(27) $E(u_i y^*) = E(u y^*) = 0$ for all i,

i.e., the errors are uncorrelated with the true income, y^*, but correlated with the income observed, y.

Second, we assume that the errors generating the variations in the coefficients β_i, β, γ_i, and γ as well as the disturbances of the first-order conditions (9) and (10) have zero expectations and are uncorrelated with the true income, i.e.,

(28) $E\varepsilon_i = E\varepsilon = Ev_i = Ev = Ew_i = Ew = 0$ for all i,

(29) $E(\varepsilon_i y^*) = E(\varepsilon y^*) = E(v_i y^*) = E(v y^*) = E(w_i y^*) = E(w y^*) = 0$ for all i.

Third, we assume

(30) u_i, v_i, ε_i, w_i, u, v, ε, w are mutually uncorrelated and uncorrelated with p_j, P, for all i, j.

By application of eqs. (2) and (3) this implies

(31) u_i, v_i, w_i, u, v, w are uncorrelated with β_i^*, β^*; and $E(\varepsilon_i \beta^*) = E(\varepsilon \beta_i^*) = 0$ for all i,

(32) u_i, ε_i, w_i, u, ε, w are uncorrelated with γ_i^*, γ^*; and $E(v_i \gamma^*) = E(v \gamma_i^*) = 0$, for all i.

Eqs. (29) together with (2), (3), (11) and (12) imply

(33) v_i, v, ε_i, ε, w_i, w are correlated with x_i^*, s^*.

I.e., even if the v's, ε's and w's are uncorrelated with the true income y^*, they are, in general, correlated with its components, true consumption and true saving. Accordingly, they are also correlated with the observed consumption and observed saving (compare eqs. (4)-(5)). It is interesting, however, to notice that the v's, ε's and w's are uncorrelated with the observed income, since they are uncorrelated with all of its components as given in eq. (8). On the other hand, they are correlated with its components based on eq. (7). The situation can be summarised as follows:

	u_i, u	v_i, v, ε_i, ε, w_i, w
y^*	Uncorrelated	Uncorrelated
y	Correlated	Uncorrelated
x_i^*, s^*	Uncorrelated	Correlated
x_i, s, c	Correlated	Correlated

Generally, the expectations of δ_i and δ will be different from zero, with the assumptions made above. However, the expected values of the numerators of eqs. (17) and (18) equal zero, in view of assumptions (28) and (30). Moreover, the expectations of the δ's add to zero identically ($\Sigma E\delta_i + E\delta = 0$).

The composite errors U_i and U, being functions of all the errors and disturbances in the model (compare eqs. (19)-(20) and (19a)-(20a)), are correlated with observed income, unless the errors in consumption and saving are equal to zero ($u_i = u = 0$). This is evident from the summary table above. It is also obvious that U_i and U are correlated with observed consumption expenditure c regardless of whether u_i and u equal zero or not. Thus, estimation of the LES equations (23) by means of ordinary least squares would give rise to 'simultaneity bias' as well as 'errors-of-measurement bias', whereas application of OLS to eqs. (19a), (20a) and (22) would result in the latter sort of bias only. Of course, before taking a closer look at the problem of estimation, the identifiability of the coefficients of the model should be investigated. This is outside the scope of this note.

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