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A comparison of approximately optimal stratification given proportional allocation with other methods of stratification and allocation

By

Ib Thomsen*)

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ABSTRACT

In this article, we shall present an approximately optimal method for constructing stratum boundary points when the sample is allocated proportionally. The method is based on an equal partitioning of the cumulative of $f^{1/3}$, where f is the distribution of the stratification variable. We show that in many practical situations this technique compares favourably with approximately optimal stratification and allocation methods previously suggested.

1. Introduction

Our aim is to estimate the population mean of some quantitative characteristic Y in a finite population. The population is partitioned into L strata, and from each a simple random sample is selected. Let the h-th stratum contain N_h units with Y- values Y_{hi} $(i=1,2,..,N_h)$.

Let $N = \Sigma N_h$, and denote

$$\bar{Y}_h = N_h^{-1} \sum_{i=1}^{N_h} Y_{hi},$$

and

$$S_h^2 (Y) = (N_h - 1)^{-1} \sum_{i=1}^{N_h} (Y_{hi} - \bar{Y}_h)^2.$$

The population mean is $\bar{Y} = \Sigma \bar{Y}_h N_h /N$. We denote the sample size in stratum h by n_h and the i-th observed Y- value in stratum h by y_{hi}. The h-th stratum mean is $\bar{y}_h = n_h^{-1} \Sigma y_{hi}$, and $\bar{y}_{st} = \Sigma \bar{y}_h N_h/N$ is an unbiased estimator of $\bar{Y}$, given the Y_{hi}.

The variance of $\bar{y}_{st}$ depends on how the strata are constructed, how the sample is allocated, and whether stratification is done by means of Y or by some auxiliary covariable X. In this paper we shall give methods to construct strata such that var $(\bar{y}_{st})$ is approximately minimized given proportional allocation of the sample, i. e., $n_h = (N_h/N)\, n$, when stratification is done by means of Y, and also when X is the stratifying variable, inthe case where the regression of Y on X is assumed to be linear.

We shall take the position that the population Y- values are the values of independent identically distributed variables generated from a background distribution with density f. On this assumption the optimal construction of strata, given proportional allocation, has been determined by Dalenius [3], [4]. For construction of L strata by a choice of L-1 intermediate boundary points $p_1 < \ldots < p_{L-1}$ on the Y - scale, var (y_{st}) is minimized when the following equations are satified:

$$(1.1)\quad P_h = (\zeta_h + \zeta_{h+1}) /2,$$

where

$$\zeta_h = \int_{p_{h-1}}^{p_h} y\, f(y)\, dy / W_h,$$

with

$$(1.2)\quad W_h = \int_{p_{h-1}}^{p_h} f(y)\, dy.$$

The exact solution of these equations requires complicated interative methods. In the present paper we suggest approximate solutions, and we give an approximation formula for the variance of $\bar{y}_{st}$ when the sample is allocated proportionally and the boundary points are chosen as the approximate solution of (1.1). This approximation of the variance only depends on n, L, and the distribution of the stratifying variable. A formula of this kind has several advantages, notably,

(i) It enables us to choose n and L optimally for fixed cost, and

(ii) It makes it possible to compare the variances of stratified means when different stratification and allocation methods are used.

In Section 2, we shall show that when the sample is allocated proportionally, approximately optimal method of finding the stratum boundary points consists of forming the cumulative of $f^{1/3}$ and then partitioning the cum $f^{1/3}$ scale into equal intervals.

When the strata are constructed by means of X, the method consists of applying the cum $f^{1/3}$ rule to X. We give the variances of the stratified means when either of these rules are applied. In Section 3, these methods are compared with the well-known cum $f^{1/2}$ rule and equal allocation of the sample (i.e., $n_h = n/L$ for all h) suggested by Dalenius and Hodges [4], and further studied by Serfling [9]. We show that when stratification is done by means of Y, the ratio of the variance by the cum $f^{1/3}$ method to the variance by the Dalenius-Hodges method is never less than 1 and is independent of the number of strata (apart from the fact that the approximations applied become more accurate as L increases). When stratification is done by means of X, however, the same ratio decreases with increasing L, and it becomes smaller than 1 for sufficiently large L.

In [10], Singh recommends the use of the cum $f^{1/3}$ rule to find the stratum boundary points when stratification is done by means of X. His formula for determining n_h involves the regression coefficient and the variance of the residuals. In many practical situations these are unknown, and a simple allocation formula is needed. In Section [4], we show that proportional allocation gives a smaller variance than does equal allocation when the cum $f^{1/3}$ rule is applied to construct strata. In Section 5, we find the optimal choice of n and L for fixed cost when the cum $f^{1/3}$ rule is applied to construct strata and the sample is allocated proportionally.

2. Approximately optimal stratification with proportional allocation

Denote

$$H(Y) = \int_{-\infty}^{\infty} [f(y)]^{1/3} dy.$$

Let us confine our attention to a finite interval $[a,b]$ outside of which $f(y)$ may be assumed to be zero with negligible error. Let $p_1(y) < p_{L-1}(Y)$ be the boundary points defining a construction of L strata within the interval $[a,b]$ and set $p_0(Y) = a$, $p_L(Y) = b$. Denote

$$B_h(Y) = \int_{p_{h-1}(y)}^{p_h(y)} [f(y)]^{1/3} dy,$$

$$\sigma_h^2(Y) = \int_{p_{h-1}(y)}^{p_h(y)} y^2 \frac{f(y)}{W_h(Y)} dy - \{ \int_{p_{h-1}(y)}^{p_h(y)} y \frac{f(y)}{W_h} dy \}^2$$

and

$$\sigma^2(Y) = \int_{-\infty}^{\infty} y^2 f(y)\, dy - \{ \int_{-\infty}^{\infty} y f(y)\, dy \}^2,$$

with $W_h(Y)$ given as in (1.2). Let $f(y)$ be approximated within the h-th stratum by its mean value $\xi_h(Y)$ therein. Then the weight, variance, and $B_h(Y)$ of the h-th stratum are approximately

(2.1) $W_h(Y) \doteq \xi_h(Y) [p_h(Y) - p_{h-1}(Y)]$,

(2.2) $\sigma_h^2(Y) \doteq [p_h(Y) - p_{h-1}(Y)]^2 / 12$,

and

(2.3) $B_h(Y) \doteq \xi_h^{1/3}(Y) [p_h(Y) - p_{h-1}(Y)]$.

The model adopted in this paper seems to differ slightly from the ones suggested in [3], [4], [5], [9], and [10], in that we have assumed explicitly that the population Y - values are generated by a background distribution. This makes no difference to the mathematics, however. If we let

$$\underset{\sim}{Y} = \{Y_{hi}: \text{all } h \text{ and } i\}$$

be the set off all population Y - values, then

$$E(\bar{y}_{st} | \underset{\sim}{Y}) = \bar{Y}$$

and

$$\text{var}(\bar{y}_{st}) = E\, \text{var}(\bar{y}_{st} | \underset{\sim}{Y}) + \text{var}\, E(\bar{y}_{st} | \underset{\sim}{Y}).$$

If for convenience we write

$$\mathrm{Var}(\bar{y}_{st}) = E \ \mathrm{var}\ (\bar{y}_{st} | \underset{\sim}{Y}),$$

therefore,

$$\mathrm{var}\ (\bar{y}_{st}) = \mathrm{Var}\ (\bar{y}_{st}) \pm \mathrm{var}\ \bar{Y}.$$

Since var $\bar{Y}$ is independent of the stratification and allocation method, a discussion of how to minimize var $(\bar{y}_{st})$ centers on a similar discussion of Var $(\bar{y}_{st})$, which is essentially what previous authors have given. This permits us to concentrate on Var $(\bar{y}_{st})$, which we shall do.

Lemma 1 in the appendix tells us that

$$(2.4) \quad \mathrm{Var}\ (\bar{y}_{st}) \doteq n^{-1} \sum_{h=1}^{L} W_h(Y)\ \sigma_h^2\ (Y).$$

Inserting (2.1), (2.2), and (2.3) into (2.4), we get

$$(2.5) \quad \mathrm{Var}\ (\bar{y}_{st}) \doteq n^{-1} \sum_{h=1}^{L} B_h^3(Y)\ /12.$$

Since $\Sigma B_h(Y) = H(Y)$ is independent of the choice of boundary points, (2.5) is a minimum when $B_h(Y)$ is a constant for all h, i.e., $B_h(Y) = H(Y)/L$. In that case

$$(2.6) \quad \mathrm{Var}\ (\bar{y}_{st}) = H^3(Y)/(12L^2n).$$

Usually, the stratification cannot be carried out by means of the study variable. We **therefore** turn to the more realistic situation where the stratification is done by means of an auxiliary variable X. We shall suppose that the regression of Y on X is linear, that is

$$(2.7) \quad Y_{hi} = \alpha + \beta X_{hi} + U_{hi},$$

where the U_{hi} are independent of each other and of X_{hi}, and where $E(U_{hi})=0$, var $(U_{hi})=\sigma^2$. Dalenius and Hodges [3] give equations for intermediate stratum boundary points on the X - scale which make Var $(\bar{y}_{st})$ a minimum for a proportional allocation of the sample. The solution consists in applying rule (1.1) to X. We correspondingly apply the cum $f^{1/3}$ rule given above to X. Formula (2.4) then gives

$$(2.8) \quad \mathrm{Var}\ (\bar{y}_{st}) \doteq n^{-1} \sum_{h=1}^{1} W_h(X) \left[\beta^2 \sigma_h^2(X) + \sigma^2\right]$$

$$\doteq n^{-1}\{\beta^2 H^3(X)/(12L^2) + \sigma^2\}.$$

Under assumptions (2.7), $\beta^2 = \rho^2\sigma^2(Y)/\sigma^2(X)$ and $\sigma^2 = (1-\rho^2)\ \sigma^2(Y)$, where ρ is the correlation coefficient between X and Y. Then by (2.8) it follows that

$$\mathrm{Var}\ (\bar{y}_{st}) \doteq n^{-1}\sigma^2(Y)\ \{\rho^2 H^3(X)/[12\ L^2\sigma^2(X)] + (1-\rho^2)\}$$

$$(2.9) \qquad \doteq n^{-1}\sigma^2(Y)\ \{H^{*}(X)\rho^2/L^2 + (1-\rho^2)\},$$

where

$$H^{*}(X) = H^3(X)/[12\ \sigma^2(X)].$$

Formulas (2.6) and (2.9) parallel results derived by Serfling [9] using different methods of stratification and allocation. These methods will be compared with the methods given above in the next section.

3. A comparison of the methods given in Section 2 with the Dalenius-Hodges stratification method and equal allocation.

The following method of stratification is studied and recommended in several books and articles [1 ; pp 128-123], [2], [3], [5], [7], [8 ; p. 105], [9]. First the cumulative of $f^{1/2}$ is formed, and then the $f^{1/2}$ scale is partitioned into equal intervals. The allocation consists of taking equally many observations from each stratum. An approximation to the mean of the conditional variance of a stratified mean, $\bar{y}^{*}_{st}$, using this stratification and allocation method, is given in [9] as

$$(3.1) \quad \mathrm{Var}\ (\bar{y}^{*}_{st}) = K^4(Y)/(12\ n\ L^2),$$

where

$$K(Y) = \int_{-\infty}^{\infty} [f(y)]^{1/2} dy.$$

From (2.6) and (3.1) it follows that

$$(3.2) \quad \frac{\mathrm{Var}\ (\bar{y}_{st})}{\mathrm{Var}\ (\bar{y}^{*}_{st})} = \frac{H^3(Y)}{K^4(Y)}.$$

This ratio is independent of L apart from the fact that the approximations become more accurate as the number of strata increases. If we apply the same approach as in Section 3 in [9], we easily verify that the ratio (3.2) is invariant under a change of either location or scale From lemma 2 in the appendix (with n=4) it follows that the ratio in (3.2) is never less than 1. Let us examine this ratio for some particular distributions.

(i) Rectangular class: $f(y) = d^{-1}$ for $c \leq y \leq c + d$. We find that $K^4 = d^2$ and $H^3 = d^2$. As expected, the ratio in (3.2) equals 1.
(ii) Normal class: $f(y) = (2\Pi)^{-1/2}\sigma^{-1}\exp\left[-(y-\xi)^2/2\sigma^2\right]$.
Here $K^4 = 8\Pi\sigma^2$ and $H^3 = 2\Pi\sigma^2 3^{3/2}$. The ratio equals $3^{3/2}/4 \doteq 1.3$.
(iii) Exponential class. $f(y) = \lambda e^{-\lambda y}$ for $y \geq 0$. Now, $K^4 = 16\lambda^{-2}$ and $H^3 = 27\lambda^{-2}$. The ratio equals $27/16 \doteq 1.7$.

It follows that when stratification is done by means of Y, the method suggested in Section 2 results in a non-trivial increase in mean conditional variance over the cum $f^{1/2}$ method and equal allocation. The increase is independent of L. However, this is not so in the more realistic situation where stratification is done by means of the auxiliary variable X. Serfling [9] gives the following result when strata are constructed by applying the cum $f^{1/2}$ method to X, and when $n_h = n/L$:

$$(3.3)\quad \text{Var}\ (\bar{y}_{st}^{*}) \doteq n^{-1}\sigma^2(Y)\ \{k_x\rho^2/L^2 + k_x^{*}\ (1-\rho^2)\},$$

where

$$k_x = K^4(X)/\left[12\sigma^2(X)\right],$$

$$K_x^{*} = \int_{-\infty}^{\infty} g(x)^{1/2}dx \int_{-\infty}^{\infty} g(x)^{3/2}dx,$$

and g is the density of X. Now by the Schwarz inequality we have that

$$(3.4)\quad k_x^{*} = \int_{-\infty}^{\infty}\left[g(x)^{1/4}\right]^2 dx \int_{-\infty}^{\infty}\left[g(x)^{3/4}\right]^2 dx \geq \{\int_{-\infty}^{\infty} g(x)dx\}^2 = 1.$$

From (3.3), (3.4), and (2.9), the somewhat surprising result follows that the difference between Var $(\bar{y}_{st}^{*})$ and Var $(\bar{y}_{st})$ decreases as L increases, and that for sufficiently large L, Var $(\bar{y}_{st})$ is smaller than Var $(\bar{y}_{st}^{*})$ when $\rho \neq 0$. Let us see how Var $(\bar{y}_{st}^{*})$ and Var $(\bar{y}_{st})$ decrease with L for particular distributions of X and different values of ρ.

(i) Rectangular class:

In this case Var $(\bar{y}_{st}^{*})$ = Var $(\bar{y}_{st})$ for all L.

(ii) Normal class:

Serfling [9] gives $k_x = 2\Pi/3 \doteq 2{,}09$, $k_x^{*} \doteq 2/3^{\frac{1}{2}} = 1{,}16$.
We find that $H^{*} = \Pi 3^{\frac{1}{2}}/2 = 2{,}67$ In table 1 below we have tabulated V_1 = Var $(\bar{y}_{st}^{*})$ / $\left[n\ \sigma^2(Y)\right]$ and V_2 = Var $(\bar{y}_{st})/\left[n\sigma^2(Y)\right]$ for $\rho = 0.99$ and for $\rho = 0.90$.

Table 1

	ρ = 0.99		ρ = 0.90	
L	V_1	V_2	V_1	V_2
2	0.54	0.68	0.64	0.73
3	0.25	0.31	0.41	0.43
4	0.15	0.18	0.33	0.33
5	0.11	0.13	0.29	0.28
6	0.08	0.09	0.27	0.25
7	0.07	0.07	0.26	0.23

(iii) Exponential class:

In this case, Serfling [9] gives $k_x = k_x^* = 1.33$. Our (2.9) gives $H^* = 2.25$. In Table 2 below, V_1 and V_2 are tabulated for the exponential class for $\rho = 0.99$ and for $\rho = 0.90$.

Table 2

	ρ = 0.99		ρ = 0.90	
L	V_1	V_2	V_1	V_2
2	0.35	0.57	0.52	0.65
3	0.17	0.27	0.37	0.39
4	0.11	0.16	0.32	0.30
5	0.08	0.11	0.29	0.26
6	0.06	0.08	0.28	0.24
7	0.05	0.07	0.28	0.23

We see from the tables that even with a small number of strata the difference between Var $(\bar{y}_{st}^*)$ and Var $(\tilde{y}_{st})$ is negligible or even negative.

4. A comment on Singh's method for stratification and allocation.

When stratification is done by means of X, Singh [10] suggests applying the cum $f^{1/3}$ rule for the construction of the strata, but his formula for n_h involves β and σ^2, which are unknown in general. Therefore we cannot compare the Singh stratification and allocation method with the methods proposed in Section 2. In this section we shall find mean conditional variance of the stratified mean, say $\bar{y}^*$, when the cum $f^{1/3}$ rule is used to construct strata and the sample is allocated equally over the strata. Lemma 1 of the appendix shows this to be

$$\text{Var}(\bar{y}^{*}) \doteq n^{-1}L\sum_{h=1}^{L} W_h^2\,\sigma_h^2(Y)$$

$$\doteq n^{-1}L\{\sum_{h=1}^{L} \beta^2\xi_h(X)\,[p_h(X)-p_{h-1}(X)]^4/12 + \sigma^2\sum_{h=1}^{l} W_h^2(X)\}$$

$$(3.5)\qquad \doteq n^{-1}L\{\sum_{h=1}^{L} \beta^2 B_h^3(X)\xi_h(X)\,[p_h(X)-p_{h-1}(X)]/12$$

$$+\ \sigma^2\sum_{h=1}^{L} \xi_h^2(X)\,[p_h\ V\ (X)-p_{h-1}(X)]^2\}$$

$$\doteq n^{-1}L\{\frac{\beta^2H^3(X)}{12\ L^3} + \sigma^2\sum_{h=1}^{L} B_h(X)\xi_h^{5/3}(X)\,[p_h(X)-p_{h-1}(X)]\}$$

$$\doteq n^{-1}\{\beta^2H^3(X)/(12L^2) + \sigma^2\int_{-\infty}^{\infty}g(x)^{1/3}dx\int_{-\infty}^{\infty}g(x)^{5/3}dx.$$

Applying the Schwarz inequality we find

$$(3.6)\ \int_{-\infty}^{\infty}g(x)^{1/3}dx\int_{-\infty}^{\infty}g(x)^{5/3}dx = \int_{-\infty}^{\infty}[g(x)^{1/6}]^2dx\int_{-\infty}^{\infty}[g(x)^{5/6}]^2dx \geq$$

$$\{\int_{-\infty}^{\infty}g(x)dx\}^2 = 1.$$

From (3.5), (3.6), and (2.8), it follows that

$$\text{Var}(\bar{y}^{*}) \geq \text{Var}(\bar{y}_{st}),$$

at least for large enough L. Thus, the cum $f^{1/3}$ stratification rule works better when it is combined with proportional allocation than when it is combined with equal allocation.

4. Optimal choice of L and n for fixed cost.

In this Section, we shall demonstrate how (2.6) and (2.9) may be used to make an optimal choice of L and n when applying the stratification and allocation methods given in Section 2. Following [9], we assume that the cost function has the form

$$(4.1)\ C_1 = c_{o1} + c_1 n + \emptyset_1(L),$$

where we have assumed that the cost c_1 per unit is the same in all strata. The quantity c_{o1} represents overhead cost and $\emptyset_1(L)$ represents the cost of forming L strata.

Under stratification by means of Y, the mean conditional variance in (2.6) is to be minimized under condition (4.1). Applying the usual lagrange technique, we find

$$(4.2) \quad n = L\, \emptyset_1'(L)/(2c_1),$$

where $\emptyset_1^*$ denotes the deviative $\emptyset_1$, and where L may be found from

$$(4.3) \quad C_1 = c_{o1} + \tfrac{1}{2} L\, \emptyset_1'(L) + \emptyset_1(L).$$

This solution is independent of the distrbution of Y. Relations (4.2) and (4.3) are formally equal to the corresponding relations (4.3) and (4.4) in [9]. One should, however, keep in mind that c_{o1} and $\emptyset_1'(L)$ typically are smaller here than the corresponding c_o and $\emptyset'(L)$ in [9], because the estimation procedure is more costly under equal allocation used in [9], than it is under proportional allocation.

When stratification is done by means of X, we want to minimize the mean conditional variance in (2.9), subject to condition (4.1). We find that the optimal L satisfies the following equation:

$$(4.4) \quad L^3\emptyset_1'(L) + \mu L\emptyset_1'(L) - 2\mu(C_1 - c_{10} - \emptyset_1(L)) = 0,$$

where

$$\mu = H^{*}(X)\rho^2/(1-\rho^2),$$

while the optimal n is given by (4.2)

It is seen that the optimal L and n in this case depends on the distribution of X through $H^{*}(X)$.

Appendix.

When the finite population values of Y are assumed to be generated from a background distribution f, we shall prove a lemma about the conditional variances of $\bar{y}_{st}$ and $\bar{y}^{*}$ defined in Section 2 and Section 4 above.

Lemma 1: Let $\underset{\sim}{Y} = (Y_{hi}$: all h and i). Then, ignoring the finite population correction, we have that

$$\mathrm{Var}\,(\bar{y}_{st}) \equiv E\ \mathrm{var}\,(\bar{y}_{st} | \underset{\sim}{Y}) = \frac{1}{n} \sum_{h=1}^{l} W_h(y)\sigma_h^2(Y)$$

and

$$\mathrm{Var}(\bar{y}^*) \equiv E\ \mathrm{var}\ (\bar{y}^* | \underset{\sim}{Y}) \doteq Ln^{-1} \sum_{h=1}^{L} \sigma_h^2(Y) W_h^2(Y)$$

Proof: Ignoring the finite population correction, we find

$$\mathrm{var}\ (\bar{y}_{st} | \underset{\sim}{Y}) = \sum_h \frac{s_h^2(Y)}{n_h} \frac{N_h^2}{N^2}$$

$$= \frac{1}{n} \sum_h S_h^2(Y) \frac{N_h}{N}.$$

The expected mean of $S_h^2(Y)$, given N_h and given that all Y_{hi} appearing in $S_h^2(Y)$ belong to stratum h, is $\sigma_h^2(Y)$. Thus,

$$E\ \{S_h^2(Y) \frac{N_h}{N}\} = E\{\sigma_h^2(Y) \frac{N_h}{N}\} = W_h(Y) \sigma_h^2(Y),$$

which proves the first part of the lemma. The rest of the proof is similar. The second relation is an approximation formular because we make use of the approximation $W_h(Y)\{1-W_h(Y)\}/N \simeq 0$ in its proof.□

The following lemma is useful in Section 3.

Lemma 2. Let Z denote a random variable with density h, and assume that

$$\int_{-\infty}^{\infty} [h(Z)]^{\frac{1}{n-1}} dZ < \infty$$

for some n > 1. Then

$$\{E[h(Z)]^{-\frac{n-2}{n-1}}\}^{n-1} \geq \{E[h(Z)]^{-\frac{n-2}{n}}\}^n, \quad n > 1.$$

Proof: The proof is based on Hölder's inequality as stated in theorem 188 in [6,p140], from which it follows that

$$\{\int_{-\infty}^{\infty} h(Z)^{\frac{1}{n-1}} dZ\}^{\frac{n-1}{n}} \{\int_{-\infty}^{\infty} h(Z) dZ\}^{\frac{1}{n}} \geq \int_{-\infty}^{\infty} h(Z)^{\frac{2}{n}} dZ.$$

This is equivalent to

$$\{\int_{-\infty}^{\infty} h(Z)^{\frac{1}{n-1}} dZ\}^{n-1} \geq \{\int_{-\infty}^{\infty} h(Z)^{\frac{2}{n}} dZ\}^n,$$

which was to be proved. □

References

[1] Cochran, W. (1963). *Sampling Techniques.* Wiley, N.Y.

[2] Cochran, W. (1961). Comparison of methods for determining stratum boundaries. Bull. Int. Statist. Inst. **38** (2), 345-358.

[3] Dalenius, T. (1957). *Sampling in Sweden.* Almqvist & Wiksell, Stockholm.

[4] Dalenius, T. and Hodges, J. L. (1959). Minimum variance stratification. *J. Amer Statist. Ass.*, **54**, 88-101.

[5] Ekman, G. (1959). An approximation useful in univariate stratification. *J. Amer. Statist. Ass.*, **30**, 219-229.

[6] Hardy, G. H., Littlewood, J. E., and Polya, G. (1952). *Inequalities.* Cambridge University Press.

[7] Hess, I., Sethi, V. K., and Balakrishnan, T. R. (1966). Stratification: A practical investigation. *J. Amer. Statist. Ass.* **61**, 74-90.

[8] Kish, L. (1965). *Survey Sampling.* Wiley, N. Y.

[9] Serfling, R. J. (1971). Approximately optimal stratification. *J. Amer. Statist. Ass.* **63**, 1298-1309.

[10] Singh, R. (1971). Approximately optimal stratification on the auxiliary variable. *J. Amer. Statist. Ass.* **66**, 829-833.