Arbeidsnotater

STATISTISK SENTRALBYRÅ

IO 68/27 Oslo, 16 December 1968

COMPOSABLE MARKOV PROCESSES

by Tore Schweder x)

CONTENTS

x) The paper has been written in the Study Group for Population Models.

1. Introduction and summary

Many phenomena studied in the social sciences and elsewhere are complexes of more or less independent characteristics which develop simultaneously. Such phenomena may often be realistically described by time-continuous denumerable Markov processes. In order to define such a model which will take care of all the relevant a priori information, there ought to be a way of defining a Markov process as a vector of components representing the various characteristics constituting the phenomenon such that the dependences between the characteristics are represented by explicit requirements on the Markov process, preferably on its infinitesimal generator.

In this paper a stochastic process is defined to be composable if, from a probabilistic point of view, it may be regarded as a vector of distinct sub-processes.

In a composable Markov process the concept of force indenpendence between its components is defined as restrictions on the infinitesimal generator. The paper gives a set of theorems on the relations between the concepts of force independence and stochastic independence.

2. Composable processes

§ 2A. Let $Y = Y(t)$ be a stochastic process with real time T and an at most denumerable state space E. Assume that there are $p \geq 2$ spaces E_i; $i=1,\dots,p$; such that the number of elements of each space at least equals 2, and that there exists a one-to-one mapping f of E on to $\mathop{X}_{i=1}^{p} E_i$.

Definition: The process Y is a composable process with components $Y_1,\dots,Y_p$ given by $f(Y(t)) = (Y_1(t),\dots,Y_p(t))$ if and only if for each $A \subset \{1,2,\dots,p\}$ with at least 2 elements,

$$\lim_{h \downarrow 0} \frac{1}{h} P\{\bigcap_{i \in A} Y_i(t+h) \neq y_i \mid \bigcap_{i=1}^{p} Y_i(t) = y_i\} = 0$$

whenever $y_i \in E_i$; $i=1,\dots,p$; and $t \in T$.

In other words: Y is a composable process with components Y_i; $i=1,\dots,p$; if the probability that more than one component changes value during a period of length h, is of magnitude $o(h)$. If this is the case, we write

$$Y \sim (Y_1,\dots,Y_p).$$

§ 2B. The compositioning of $Y \sim (Y_1,\ldots,Y_p)$ is not necessarily unique. If $p > 2$ let $A_1,\ldots,A_r$; $2 \leq r < p$, be a partitioning of $\{1,\ldots,p\}$, i.e. if $i \neq j$ then $A_i \cap A_j = \emptyset$, $A_i \neq \emptyset$ for $i = 1,\ldots,r$, and $\bigcup_{i=1}^{r} A_i = \{1,\ldots,p\}$. We can then define $E'_j = \underset{i \in A_j}{X} E_i$ and f' as the one-to-one mapping of E on $\underset{j=1}{\overset{r}{X}} E'_j$ induced by f. In this case we consequently have

$$Y \sim (Y_1,\ldots,Y_p) \sim (Y'_1,\ldots,Y'_r)$$

where $(Y'_1(x),\ldots,Y'_r(x)) = f'(Y(x))$.

§ 2C. If $Y \sim (Y_1,\ldots,Y_p)$ is a composable Markov process such that all forces of transition $\mu_x(y;y')$ exist, then $\mu_x(y;y')$ equals zero if y and y' differ on more than one component, $y \neq y' \in \underset{i=1}{\overset{p}{X}} E_i$. This is an immediate consequence of the following definition of the forces of transition:

$$\mu_x(y;y') = \lim_{h \downarrow 0} \frac{1}{h} P\{Y(x+h) = y' \mid Y(x) = y\}.$$

3. Force independence

§ 3A. Let Y be a composable Markov process with finite state space. We shall call Y a CFMP (Composable Finite Markov Process) if for all $y, y' \in E$ such that $y \neq y'$ the force of transition $\mu_x(y;y')$ exists and is a continuous and bounded function of x on any closed interval in T.

A CFMP Y has a normal transition-probability $P_{xt}(y,y') = P\{Y(x+t) = y' \mid Y(x) = y\}$, i.e. $\lim_{t \downarrow 0} P_{xt}(y,y')$ equals 0 or 1 according as y and y' are different or equal. In this case the total force of transition $\bar{\mu}_x(y) = \sum_{y' \neq y} \mu_x(y,y') = \lim_{t \downarrow 0} \frac{1}{t}(1 - P_{xt}(y,y))$ is a continuous and bounded function of t.

§ 3B. Let $Y \sim (Y_1,\ldots Y_p)$ be a CFMP. According to § 2C only those $\mu_x(y;y')$ differ from 0 for which y and y' are equal in all but one component, say the r-th. In order to suppress superfluous arguments, we let

$$\mu_x(y;y') = \gamma_x^r(y;y'_r)$$

where y'_r is the r-th component of y'.

Definition: The component Y_q is force independent of the component Y_r if and only if $\gamma_x^q(y;y_q')$ is a constant function of the r-th component y_r of y for all $x \in T$, $y_q' \in E_q$ and $y_i \in E_i$, $i \neq r$.

The relation "force independent of" is neither symmetric, reflexive, nor transitive.

Y_j will be said to be force dependent on Y_q when it is not force independent of Y_q. When Y_j is force dependent on exactly $Y_{i_1},\ldots,Y_{i_k}$ it is convenient to write

$$\gamma_x^j(y;y_j') = \lambda_x^j(y_{i_1},\ldots,y_{i_k};y_j')$$

§ 3C. We shall elucidate the relation between force independence and stochastic independence by proving some theorems.

Theorem 1. Let $Y \sim (Y_1,Y_2)$ be a CFMP. If Y_1 is force independent of Y_2, then $\{Y_1(x)\}$ is a Markov process with forces of transition $\lambda_x^1(y_1;y_1')$.

Proof. Assume that $[x^0,x] \subset T$. Let N be the number of transitions in $[x^0,x]$, let $Y^0 = Y(x^0)$, let X^k be the time of the k-th transition after x^0 and let Y^k be the value of Y immediately after the k-th transition. Let $(\Omega,\mathcal{A},P)$ be the probability space in which the process Y is defined. On the basis of $(\Omega,\mathcal{A},P)$ we can construct a probability space $(\mathcal{W},\mathcal{B},Q)$ for the random variable $W = (Y^0,N,X^1,Y^1,X^2,Y^2,\ldots,X^N,Y^N)$, with a natural σ-finite measure σ over $(\mathcal{W},\mathcal{B})$ constructed by means of the counting measure and lebesque measure. (See Albert, 1962, p. 731. Since our process does not have stationary transition probabilities, $\mathcal{W}$ differs (inessentially) from the space Albert constructs.)

The points w of $\mathcal{W}$ have the form $w = (y^0,n,x^1,y^1,x^2,y^2,\ldots,x^n,y^n)$ where $y^i \in E$; $y^i \neq y^{i+1}$; $i=1,\ldots,n$; $x^0 < x^1 < \ldots < x^n < x$, $n \geq 0$.

Then a stochastic process Y is a Markov process with forces of transition $\mu_t(y,y')$ fulfilling the regularity conditions previously mentioned if and only if Q is absolutely continuous with respect to σ and the Radon-Nikodym derivative $f_q = \frac{dQ}{d\sigma}$ is given by the formula

$$f_q(w) = p_{y^0} \prod_{i=1}^{n} \exp\{-\int_{x^{i-1}}^{x^i} \bar{\mu}_t(y^{i-1})dt\}\mu_{x^i}(y^{i-1},y^i) \exp\{-\int_{x^n}^{x} \bar{\mu}_t(y^n)dt\},$$

where $p_{y^0} = Pr(Y^0 = y^0) = P\{Y(x^0) = y^0\}$. (The product $\prod_{i=1}^{n}$ is interpreted as 1 if n equals 0.)

If this is the case, N is finite with probability 1 (Hoem, 1968) and f_q is a density.

Assume now that $Y \sim (Y_1, Y_2)$ is a CFMP such that Y_1 is force independent of Y_2. By the definition of composability, a transition of the process is with probability one either a transition of Y_1 or of Y_2. Consequently, it is possible to define random variables $W_1 = (Y_1^0, N_1, X_1^1, Y_1^1, X_1^2, Y_1^2, \ldots, X_1^{N_1}, Y_1^{N_1})$ and $W_2 = (Y_2^0, N_2, X_2^1, \ldots, X_2^{N_2}, Y_2^{N_2})$ which are the W-variables connected with the Y_1 and the Y_2 process, respectively. Both W_1 and W_2 are measurable functions of W, in particular $N_1 + N_2 = N$, $Y^0 = (Y_1^0, Y_2^0)$.

W_1 is defined over the σ-finite measure space $(\mathcal{W}_1, \mathcal{B}_1, \sigma_1)$ induced by $(\mathcal{W}, \mathcal{B}, \sigma)$, and has a probability measure Q_1 which is obviously absolutely continuous with respect to σ_1. Our aim is to show that $f_{Q_1} = \frac{dQ_1}{d\sigma_1}$ is given by

$$f_{Q_1}(w_1) = p_{y_1^0} \prod_{i=1}^{n_1} \exp\{-\int_{x_1^{i-1}}^{x_1^i} \bar{\lambda}_t^1(y_1^{i-1})dt\} \lambda_{x_1^i}^1(y_1^{i-1}, y_1^i) \exp\{-\int_{x_1^{n_1}}^{x} \bar{\lambda}_t^1(y_1^{n_1})dt\},$$

where $\bar{\lambda}_t^1(y_1)$ is the total force of transition for Y_1.

Let $A \in \mathcal{B}_1 \cap \{N_1 = n_1\}$. Then

$$Q_1(A) = Q(\{w | w_1 \in A\}) = \int_{\{w | w_1 \in A\}} f_Q(w)\, \sigma(dw).$$

By the force independence we have for $y = (y_1, y_2)$ the relation $\bar{\mu}_t(y) = \bar{\lambda}_t^1(y_1) + \bar{\gamma}_t^2(y_1, y_2)$. Fubini's theorem then gives

$$Q_1(A) = \int_{A \times \mathcal{W}_2} f_Q(w)\, \sigma(dw) =$$

$$= \int_A p_{y_1^0} \prod_{i=1}^{n_1} \exp\{-\int_{x_1^{i-1}}^{x_1^i} \bar{\lambda}_t^1(y_1^{i-1})dt\} \lambda_{x_1^i}^1(y_1^{i-1}, y_1^i) \exp\{-\int_{x_1^{n_1}}^{x} \bar{\lambda}_t^1(y_1^{n_1})dt \cdot$$

$$\int_{\mathcal{W}_2} p_{y_2^0} \prod_{j=1}^{n} \exp\{-\int_{x^{j-1}}^{x^j} \bar{\gamma}_t^2(y^{j-1})dt\} \prod_{i=1}^{n_2} \gamma_{x_2^i}^2(y_1^{j(i)}, y_2^{i-1}, y_2^i) \cdot$$

$$\exp\{-\int_{x^n}^{x} \bar{\gamma}_t^2(y^n)dt\}\, \sigma_2(dw_2)\, \sigma_1(dw_1)$$

$$= \int_A p_{y_1^0} \prod_{i=1}^{n_1} \exp\{-\int_{x_1^{i-1}}^{x_1^i} \bar{\lambda}_t^1(y_1^{i-1})dt\} \lambda_{x_1^i}^1(y_1^{i-1}, y_1^i) \exp\{-\int_{x_1^{n_1}}^{x} \bar{\lambda}_t^1(y^n)dt\}\, \sigma_1(dw_1)$$

$$= \int_A f_{Q_1}(w_1)\, \sigma_1(dw_1).$$

We have written $y_1^{j(i)}$ for the value of $Y_1(x_2^i)$ and $p_{y_2^0} = P(Y_2(x^0) = y_2^0 | Y_1(x^0) = y_1^0)$. □

Let $Y \sim (Y_1, \ldots, Y_p)$ be a CFMP and let A be a nonempty subset of $\{1, \ldots, p\}$. Assume that Y_j is force independent of Y_k for each $j \in A$ and $k \notin A$. If Y_1' is the vector containing $\{Y_j | j \in A\}$ and Y_2' is the vector containing the rest of the components, then $Y \sim (Y_1', Y_2')$ is a compositioning of Y such that Y_1' is force independent of Y_2'. In this case Y_1' is a Markov process which develops independently of Y_2'.

Theorem 2. Let $Y \sim (Y_1, Y_2)$ be a CFMP. Then Y_1 is force independent of Y_2 if and only if, for all $t > 0$ and x, $x + t \in T$, $Y_1(x)$ and $Y_2(x)$ are stochastically independent, given $Y_1(x)$.

Proof. Let $y_1, y_1' \in E_1, y_2 \in E_2$, $t > 0$ and x, $x + t \in T$. Assume that Y_1 is force independent of Y_2. Y_1 is then a Markov process by theorem 1, and for $h \in \langle 0, t \rangle$ we have

$$R(y_2) = Pr(Y_1(x+t) = y_1' | Y_1(x) = y_1, Y_2(x) = y_2)$$

$$= \sum_{y \in E_1} Pr(Y_1(x+t) = y_1' | Y_1(x+h) = y)\, Pr(Y_1(x+h) = y | Y_1(x) = y_1, Y_2(x) = y_2)$$

$$= \sum_{y \neq y_1} Pr(Y_1(x+t) = y_1' | Y_1(x+h) = y)\, \lambda_x^1(y_1, y) h$$

$$+ \; Pr(Y_1(x+t) = y_1' | Y_1(x+h) = y_1)(1 - \bar{\lambda}_x^1(y_1) h) + o(h)$$

Consequently

$$R(y_2) = \lim_{h \downarrow 0} Pr(Y_1(x+t) = y_1' | Y_1(x+h) = y_1)$$

which is independent of y_2 and this is equivalent to the stochastic independence of $Y_1(x+t)$ and $Y_2(x)$, given $Y_1(x)$.

Assume conversely that

$$Pr(Y_1(x+t) = y_1' | Y_1(x) = y_1, Y_2(x) = y_2) = Pr(Y_1(x+t) = y_1' | Y_1(x) = y_1).$$

For $y_1' \neq y_1$ we have by definition

$$\gamma_x^1(y_1, y_2; y_1') = \lim_{t \downarrow 0} \frac{1}{t} Pr(Y_1(x+t) = y_1', Y_2(x+t) = y_2 | Y_1(x) = y_1\ Y_2(x) = y_2) =$$

$$= \lim_{t \downarrow 0} \frac{1}{t} Pr(Y_1(x+t) = y_1' | Y_1(x) = y_1, Y_2(x) = y_2) - \lim_{t \downarrow 0} \frac{1}{t} Pr(Y_1(x+t) = y_1', Y_2(x+t) \neq y_2 | Y_1(x) = y_1, Y_2(x) = y_2).$$

By the composability, the last term equals 0, and by the assumption above we get

$$\gamma_x^1(y_1,y_2;y_1') = \lim_{t\downarrow 0} \frac{1}{t}\Pr(Y_1(x+t) = y_1' | Y_1(x) = y_1)$$

which is independent of y_2. □

We shall now show that complete mutual force independence of all components is equivalent to their stochastic independence.

Theorem 3. Let $Y \sim (Y_1,\ldots,Y_p)$ be a CFMP. Then $Y_1,\ldots,Y_p$ are stochastically independent Markov processes if and only if each component is force independent of all the others.

Proof. Assume first that $Y_1,\ldots,Y_p$ are stochastically independent Markov processes. Let $P_{xt}^j(y_j,y_j')$ be the transition probabilities of Y_j, and let $y_i,y_i' \in E_i$; $i=1,\ldots,p$ be such that $y_q \neq y_q'$ and $y_j = y_j'$; $j\neq q$. Finally, let $y = (y_1,\ldots,y_p)$ and $y' = (y_1',\ldots,y_p')$. By the assumption,

$$\gamma_x^q(y_1,\ldots,y_p;\, y_q') = \lim_{t\downarrow 0} \frac{1}{t} \prod_{j=1}^{p} P_{xt}^j(y_j,y_j')$$

$$= \lim_{t\downarrow 0} \frac{1}{t} P_{xt}^q(y_q,y_q') \cdot \lim_{t\downarrow 0} \prod_{j\neq q} P_{xt}^j(y_j,y_j)$$

which is independent of y_j for all $j\neq q$ since $P_{xt}^j(y_j,y_j)$ tends to 1 as t tends to 0.

Assume conversely complete mutual force independence of the components $Y_1,\ldots,Y_p$. By theorem 1 all Y_q are Markov processes with forces of transition $\lambda_x^q(y_q,y_q')$ and corresponding transition probabilities $P_{xt}^q(y_q,y_q')$. These transition probabilities determine a new set of transition probabilities

$$P_{xt}'(y,y') = \prod_{j=1}^{p} P_{xt}^j(y_j,y_j')$$

which belong to a CFMP Y' with stochastically independent components. Since however, the two CFMP's Y' and Y have common forces of transition, they must have identical transition probabilities and hence the Markov processes $Y_1,\ldots,Y_p$ must be stochastically independent. □

The following extention of the theorem is obvious and needs no special proof.

Corollary. Let $Y \sim (Y_1,\ldots,Y_p)$ be a CFMP and let $A_1,\ldots,A_r$ be a partitioning of $\{1,\ldots,p\}$. Then the vectors Y_j' consisting of $\{Y_i \mid i \in A_j\}$; $j = 1,\ldots,r$; are stochastically independent Markov processes if and only if Y_j and Y_k are mutually force independent whenever j and k belong to different A-s.

Complete mutual force independence is not necessary however, for some components to be stochastically independent.

Theorem 4. If $Y \sim (Y_1,Y_2,Y_3)$ is CFMP such that both Y_1 and Y_2 are force independent of Y_3, then Y_1 and Y_2 are stochastically independent Markov processes if and only if they are mutually force independent.

Proof. Let $Y' = (Y_1,Y_2)$. Then $Y \sim (Y',Y_3)$ and Y' is force independent of Y_3. By theorem 1, $Y' = (Y_1,Y_2)$ is a CFMP with forces of transition

$$\gamma_x^1(y_1,y_2;y_1') = \mu_x(y,y'),\ \text{where } y' = (y_1',y_2,y_3),\ \text{and}$$

$$\gamma_x^2(y_1,y_2;y_2^0) = \mu_x(y,y^0),\ \text{where } y^0 = (y_1,y_2^0,y_3).$$

The equivalence then follows from theorem 3. □

By introducing the relation $\prec$ defined below, we obtain an interesting ordering of the components.

Definition: The binary relation $\prec$ between components is defined as follows:

(i) If Y_j is force dependent on Y_i, then $Y_i \prec Y_j$.

(ii) $\prec$ is transitive and reflexive.

We shall say that Y_k is a predecessor of Y_j whenever $Y_k \prec Y_j$. By this concept we get the following extension of theorem 4.

Corollary to theorem 4. Let $Y \sim (Y_1,\ldots,Y_p)$ be CFMP. If the components Y_r and Y_s have no common predecessors, then Y_r and Y_s are stochastically independent random processes.

Proof. Define $A_k = \{i \mid Y_i \prec Y_k\}$. Since $\prec$ is reflexive, $k \in A_k$. The antecedent in the corollary is equivalent to $A_r \cap A_s = \emptyset$. Let now Y' be the vector consisting of the components $\{Y_i \mid i \in A_r\}$, and let Y'' be the vector consisting of $\{Y_i \mid i \in A_s\}$. If $A_r \cup A_s = \{1,\ldots,p\}$ then $Y \sim (Y',Y'')$ where Y' and Y'' are mutually force independent by construction. Consequently Y_r and Y_s are stochastically independent because Y' and Y'' are. If, however, $A = \{1,\ldots,p\} - (A_r \cup A_s) \neq \emptyset$ then define Y''' to be the vector consisting of the components $\{Y_i \mid i \in A\}$. By our construction, we have $Y \sim (Y',Y'',Y''')$ where Y' and Y'' are both force independent of Y'''. The corollary is now obtained by theorem 4. □

Note that Y_r and Y_s need not be Markov processes, even if they have no common predecessors. (See example 1 of § 5 below.)

4. Conditional Markov processes

§ 4A. Starting with a CFMP it is sometimes possible to construct new Markov processes by conditioning. Assume for example that $Y \sim (Y_1,Y_2)$ is a CFMP with time space $T = [0,\infty\rangle$ and with the property that there exists a state 1 say in E_2 such that $\Pr(Y_2(0) = 1) = 1$. Let $(\Omega,\mathcal{B},P)$ be the canonical probability space defining Y, (Dynkin, 1965, p. 85), i.e. every sample point ω of Ω represents a unique sample path $y(t,\omega) = (y_1(t,\omega), y_2(t,\omega))$ with $y_2(0,\omega) = 1$. Connect to each ω in Ω a $\omega^* = g(\omega)$ which is the sample point representing the terminating sample path $y_1(t,\omega) = y^*(t,\omega^*)$; $t \in [0,D(\omega^*)\rangle$, where $D(\omega^*)$ is the time of first departure from state 1 for $y_2(t,\omega)$. $\Omega^* = g(\Omega)$ is then a sample space to which there correspond a probability space $(\Omega^*,\mathcal{B}^*,P^*)$ where $\mathcal{B}^*$ may be taken as the largest σ-algebra such that g is measurable, and $P^* = Pg^{-1}$. This probability space determines a Markov process Y^* with state space E_1, time space $[0,\infty\rangle$, terminal time D, forces of transition $\gamma_x^*(y_1,y_1') = \gamma_x^1(y_1,1;y_1')$, and transition probabilities

$$P_{xt}^*(y_1,y_1') = P(Y_1(x+t) = y_1', Y_2(\xi) = 1 \text{ for } x<\xi\leq x+t \mid Y_1(x) = y_1, Y_2(x) = 1)$$

The truth of this is seen by elementary conditional probability.

§ 4B. Consider the probability $P(Y^*(x) = y \mid \bigcap_{i-1}^{n} Y^*(x_i) = y_i \cap D>\tau)$ for $0<x_1<\ldots<x_n<x<\tau$, and $y, y_i \in E_1$. This probability satisfies the Markov condition in the sense that for all $0<x_1<\ldots x_n<x<\tau$, $P(Y^*(x) = y \mid \bigcap_{i-1}^{n} Y^*(x_i) = y_i \cap D>\tau)$ $= P(Y^*(x) = y \mid Y^*(x_n) = y_n \cap D>\tau)$. These conditional probabilities therefore are transition probabilities for a Markov process with time space $[0,\tau]$ and with forces of transition $\gamma_x^*(y_i,y_i')$. We shall denote this process the conditional Markov process, given $Y_2 = 1$. As will be seen in the examples below, this conditional process may have a structure which is much simpler and more informative than the structure of the process from which it is determined.

5. Examples

Example 1. Let us consider a queuing model described by Khintchine (1960, p. 82). Calls arrive in a telephone central with R lines, $L_1 \dots, L_R$, according to a Poisson process with parameter λ. The service pattern is as follows: If at time x a call arrives and the lines $L_1, L_2, \dots, L_{k-1}$ are busy while L_k is free ($1 \leq k \leq R$), this call is transferred via L_k. If all R lines are busy, the call is lost.

Assume that the conversation periods are stochastically independent with a common exponential distribution with parameter 1, and that they are stochastically independent of the incoming stream of calls.

Define the random variables

$$Y_i(x) = \begin{cases} 0 \text{ if } L_i \text{ is free at time } x, \\ 1 \text{ if } L_i \text{ is busy at time } x, \end{cases} \quad i = 1, \dots, R.$$

Obviously the stochastic process $Y(x) = (Y_1(x), \dots, Y_R(x))$ is a composable finite Markov process with forces of transition given by

$$\gamma_x^k(y_1, \dots, y_R; y_k') = \begin{cases} 0 \text{ for } y_k' = 1 \text{ and } y_k = 0 \text{ and } y_1 = 0, \text{ or } y_2 = 0, \dots \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{or } y_{k-1} = 0, \\ \lambda \text{ for } y_k' = 1 \text{ and } y_k = 0 \text{ and } y_j = 1;\ j = 1, \dots, k-1, \\ 1 \text{ for } y_k' = 0 \text{ and } y_k = 1. \end{cases}$$

Consequently Y_k is force dependent on $Y_1, \dots, Y_k$, and force independent of $Y_{k+1}, \dots, Y_R$. When R = 4 we can draw a picture of this structure as in figure 1 where an arrow from Y_j to Y_k indicates that Y_k is force dependent of Y_j.

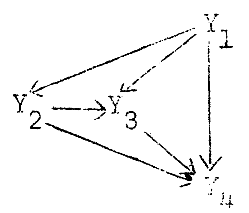

Fig. 1.

For $K < R$ we may define $Y' = (Y_1, \dots, Y_K)$ and $Y'' = (Y_{K+1}, \dots, Y_R)$. Thus $Y \sim (Y', Y'')$ where Y' is force independent of Y''. Consequently Y' is a Markov process by theorem 1 - something which is also self-evident.

Khintchine (1960, p. 83) has shown that Y_2 is <u>not</u> a Markov process despite the fact that both $Y' = (Y_1, Y_2)$ and Y_1 are.

If $R = \infty$ we get a composable Markov process $Y = (Y_1, Y_2, \dots)$ with an infinity of components. (See § 6B.)

Example 2. Suppose that one wishes to investigate the simultaneous influence on mortality of the five diseases

Y_1: chill
Y_2: pneumonia
Y_3: bronchitis
Y_4: hypertoni
Y_5: angina pectoris

A person may have or be free from each of these diseases. A live person of age x is characterized by the vector $(Y_1(x),\ldots,Y_5(x))$, where $Y_i(x)$ equals 1 or 0 according as he has or does not have the i-th disease. If the person dies at age τ, we shall say that at age $x>\tau$ he is characterized by the vector

$$(Y_1(x),\ldots,Y_5(x)) = (Y_1(\tau),\ldots,Y_5(\tau)),$$

which in fact gives his status at death.

By introducing the component $Y_6(x)$ which equals 1 or 0 according as he is alive at age x or he has died at an age $\tau \leq x$, we may give a complete characterization of him by the vector $(Y_1(x),\ldots,Y_6(x))$

A person cannot recover from any disease nor get a new one at death. It is further natural to assume that a person cannot simultaneously get two diseases, nor can he recover from one disease in the same instant as he gets another.

$Y = (Y_1,\ldots,Y_6)$ is then a composable stochastic process with the finite state space $E = \overset{6}{\underset{i=1}{X}} \{0,1\}$.

We shall assume, possibly with some lack of realism, that Y is a Markov process.

From the moment when Y_6 first equals 0, no more transfers are possible. Consequently $Y_1,\ldots,Y_5$ are force dependent on Y_6. Conversely mortality depends on the state of health, so Y_6 is force dependent on $Y_1,\ldots,Y_5$. Although we will not give any guarantee of medical reaslism, it is probably reasonable to assume that Y_1 is force independent of $Y_2,\ldots,Y_5$; Y_2 is force dependent on Y_1 and Y_3 and independent of Y_4, Y_5. Y_3 is force dependent on Y_1 and Y_2 and independent of Y_4, Y_5; Y_4 is force independent of Y_1, Y_2, Y_3, and Y_5; and Y_5 is force dependent on Y_4 and force independent of Y_1, Y_2, Y_3.

Figure 2 gives a picture of this structure.

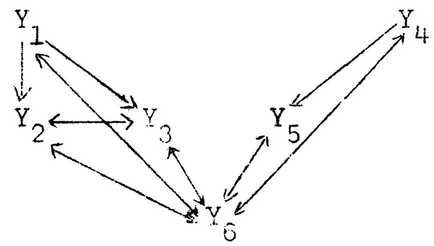

Fig. 2.

By a look at figure 2 we immediately see that for all i, j we have $Y_i \prec Y_j$. Thus all components are stochastically dependent.

If we proceed as in § 4, however, and construct the conditional Markov process $\bar{Y}$, given that $Y_6 = 1$, $\bar{Y} = (\bar{Y}_1,\ldots,\bar{Y}_5)$ is a CFMP with force dependence structure as shown i figure 3.

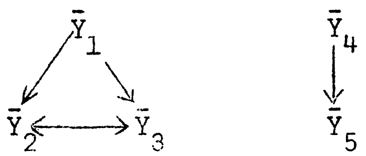

Fig. 3.

We see that the components $Y' = (\bar{Y}_1, \bar{Y}_2, \bar{Y}_3)$ and $Y'' = (\bar{Y}_4,\bar{Y}_5)$ are mutually force independent, and consequently stochastically independent Markov processes.

This illustrates a feature common to many situations where CFMP models are useful. The CFMP model describes the evolution for instance of a person, an animal, a machine, or another individual or unit which may die or stop functioning. One of the components of the CFMP indicates whether the individual (unit) is alive (functioning) or dead (out of function). In this way the rest of the components are force dependent on this particular one, and if the latter is force dependent on the others (which often is the case), then all components are stochastically dependent. If, however, we construct the conditional process described, a more interesting force dependence structure may be obtained. This structure seems to correspond to our intuitive understanding of the relations between the phenomena under consideration. In fact, we probably take into account only what happens to the individual (or unit) up to its death (or as long as it functions).

Returning to our example, let us recompose $\bar{Y}$ by letting $Y' = (\bar{Y}_1,\bar{Y}_4,\bar{Y}_5)$, $Y'' = (\bar{Y}_2, \bar{Y}_3)$, and $\bar{Y} \sim (Y',Y'')$. Because Y' is force independent of Y'', Y' is a (conditional) CFMP and since $\bar{Y}_1$ is force independent of $\bar{Y}_4$ and $\bar{Y}_5$, $\bar{Y}_1$ is a Markov process. Y'' need not, however, be a Markov process. If, on the other hand, we had recomposed $\bar{Y}$ into $\bar{Y} \sim (Y_1^0, Y_2^0)$ where $Y_1^0 = (\bar{Y}_1, \bar{Y}_5)$ and $Y_2^0 = (\bar{Y}_2,\bar{Y}_3,\bar{Y}_4)$, then neither Y_1^0 nor Y_2^0 need be Markov processes. The reason is that neither

Y_1^0 nor Y_2^0 consists of components from "the top of the force dependence tree" in figure 3.

This example throws some further light upon Markov process models in general. Let, in fact, a complicated phenomenon be described by a CFMP. This CFMP may be difficult to handle as it has too many components. The following question then arises: Is it possible to take under investigation only some part of the phenomenon which posses its main features ? Restating this question in terms of the components of the CFMP $Y \sim (Y_1,\ldots,Y_p)$, we may ask whether it is possible to recompose Y into (Y',Y''), where $Y' = (Y_{i_1},\ldots,Y_{i_q})$ describes these main features and where Y' is not too complicated for investigation ? If investigation means estimation of the probability structure of the random process Y', this may be difficult unless Y' is a Markov process. A reasonable requirement for the decompositioning of Y is therefore that Y' be such a process. If we know the force dependence structure of the process Y, we may draw a(mental or actual) picture of the "force dependence tree" as we have done in figures 1 to 3. From theorem 1 we then know that a set of components $Y_{i_1},\ldots,Y_{i_q}$ forming a "top" of this tree, if any, constitute a component $Y' = (Y_{i_1},\ldots,Y_{i_q})$ which is a Markov process.

We shall call such a component Markovian. A Markovian component of a CFMP $Y \sim (Y_1,\ldots,Y_p)$ is then by definition a component $Y' = (Y_{i_1},\ldots,Y_{i_q})$ such that for all k; $q<k\leq p$, Y_{i_k} is not a predecessor of any of the components $Y_{i_1},\ldots,Y_{i_q}$. Alternatively, if $Y \sim (Y',Y'')$ is a CFMP, then Y' is a Markovian component if Y' is force independent of Y''.

The question asked above may then be answered by looking through the possible Markovian components of Y and judging them with respect to complexity and adequacy.

6. Extension to a denumerable state space

§ 6A. In the preceding account we have considered Markov processes with a finite state space only. The theorems in § 3 are still valid however if we write CMP for CFMP everywhere, and let CMP stand for "Composable Markov Process". We define the latter concept by letting a Markov process wit a denumerable state space be a CMP if it is a composable process, and if all total forces of transition $\bar{\mu}_x(y)$ as well as all forces of transition $\mu_x(y,y')$ exist and are uniformly bounded continuous functions of x where $\bar{\mu}_x(y) = \sum_{y'\in E} \mu_x(y,y')$ holds.

§ 6B. Let us extend the concept of composability to the case of a denumerable set of components. Let Y be a random process with infinite denumerable state space. We shall call Y infinitely composable if for every integer n, Y may be composed into $Y \sim (Y_1,\ldots,Y_n)$.

By this definition, the process of example 1 with $R = \infty$ may naturally be regarded as an infinitely composable Markov process.

References

Albert, Arthur (1962). Estimating the infinitesimal Generator of a Continuous Time Finite State Markov Process. Ann.Math.Stat. 33, 727-753.

Dynkin, E. B. (1965). Markov Processes. Springer Verlag.

Hoem, Jan (1968). Application of Time-Continuous Markov Chains to Life Insurance. Memorandum dated 29.4.68. Institute of Economics, University of Oslo.

Khintchine, A. Y. (1960). Mathematical Methods in the Theory of Queuing. Griffin, London.